Lecture: Noise Theory

The topics of this lecture are:
- Noise signal in the time domain
- Analysis in the frequency domain, spectral power density of the noise signal
- Spectral power density of diode leakage current (shot noise)
- Spectral power density of thermal noise in a resistor
- Photodiode and its signal-to-noise ratio

Noise

Electronic noise is our perception of fluctuations in current flow or voltage. To calculate the amplitude of the noise signal, it is important to know the source of the noise and describe it mathematically (the input quantity), as well as to know the transfer function of the measuring device (the circuit).

The transfer function can be represented as a frequency-dependent complex function H(s) (the ratio of Fourier- or Laplace-transformed signals), provided that the measuring device (the circuit) is time-invariant.

$$H(s) = \frac{V_{\mathrm{out}}(s)}{V_{\mathrm{in}}(s)}$$

The circuit can also be represented using an impulse response. Representation via the impulse response is suitable when the measuring device changes over time—for example, when it is switched on at a specific moment.

The **impulse response** is the response to a Dirac impulse with unit area at the input. It should not be confused with the **step response**, which is the response to a step function at the input.

The impulse response can be derived by taking the time derivative of the step response. Conversely, the step response is the integral of the impulse response.

The impulse response is also the inverse Laplace or Fourier transform of the transfer function $V_{\mathrm{out}}(s)/V_{\mathrm{in}}(s)$ because the Laplace or Fourier transforms of a Dirac impulse are equal to 1.

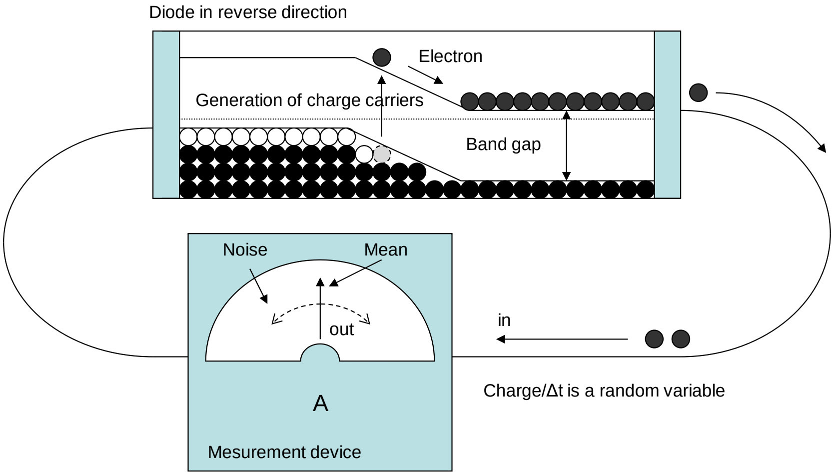


Figure1: ***Electronic noise*** *is our perception of fluctuations in current flow or voltage.* ***Source:*** *The generation of electron–hole pairs is a random process characterized by a mean value μ and a standard deviation σ. Fluctuations in the current flow are amplified by the measuring device.*

Figure 2 represents the measuring device as an ammeter (or galvanometer). This device has an output quantity, shown here as the pointer angle (φ); alternatively, the output quantity may also be a voltage or a current.

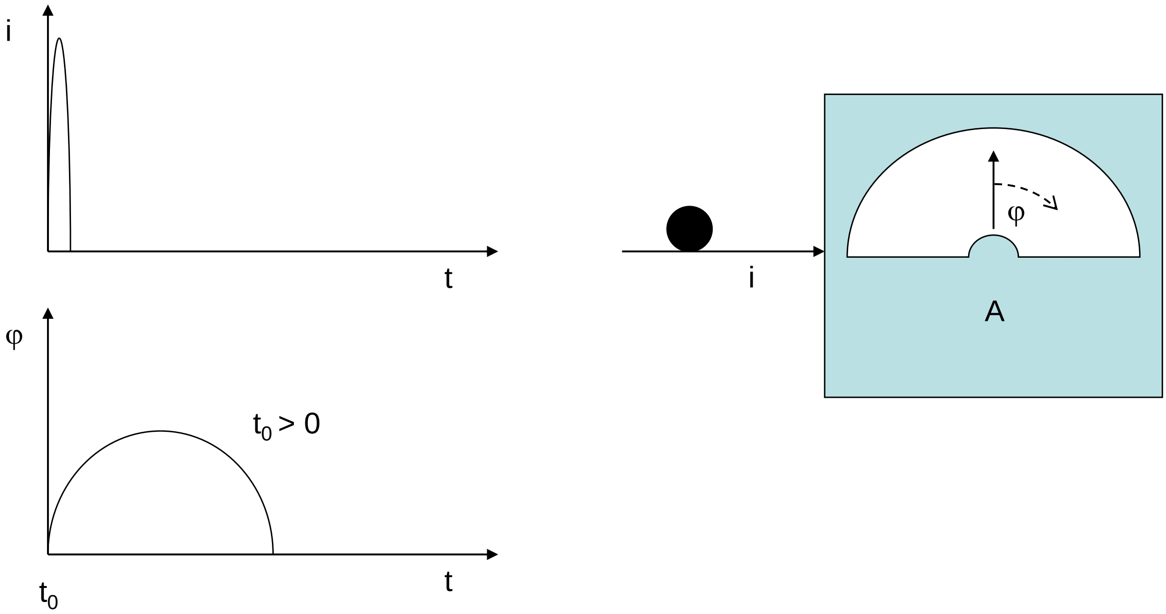


Figure2: Impulse response of the measuring device.

The output quantity φ has a specific time dependence when a current or voltage impulse with unit area (1 A·s /1 V·s) is applied at the input.

Any arbitrary input function can be represented as a sum of Dirac impulses with different amplitudes. The output is then likewise the sum of the individual impulse responses (Figure 2, left) multiplied by the corresponding amplitudes (Figure 3). The assumption here is that the system is linear.

Now let us assume that the input signal is a noise signal.

We try to calculate the output signal φ(t) at time t.

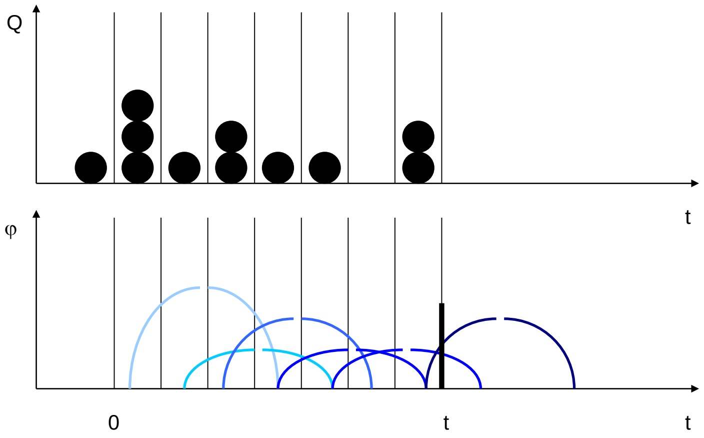


Figure3: *Any arbitrary input function can be represented as a sum of Dirac impulses with different amplitudes. The output is likewise the sum of the individual impulse responses multiplied by the corresponding amplitudes.*

We discretize time by defining time intervals $t_{i+1} - t_i = \Delta t$.

The noise-affected input signal has an area (amplitude) Qi in each interval. In the case of current noise, the unit of the amplitude is A·s; in the case of voltage noise, V·s. We also assume that Qi is a random variable and that all Qi values are independent of one another.

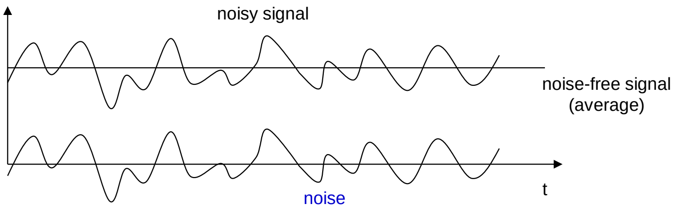
Qi is described by the mean value ⟨Qi⟩ and the variance $\langle (Q_i - \langle Q_i \rangle)^2 \rangle$ The angle brackets denote the mean value.

The mean value ⟨Qi⟩ is the noise-free signal.

We define the noise amplitude as follows:

$\langle (Q_i - \langle Q_i \rangle)^2 \rangle$

Thus, we subtract the mean value from the noise-affected amplitude. The mean value of the noise amplitude is zero, and its variance is simply $\langle q_i^2 \rangle$



The noise signal at the output of the measuring device at time t, φ(t), is approximately the sum of all impulse responses caused by the Dirac impulses in the preceding time intervals, multiplied by the noise amplitudes qi.

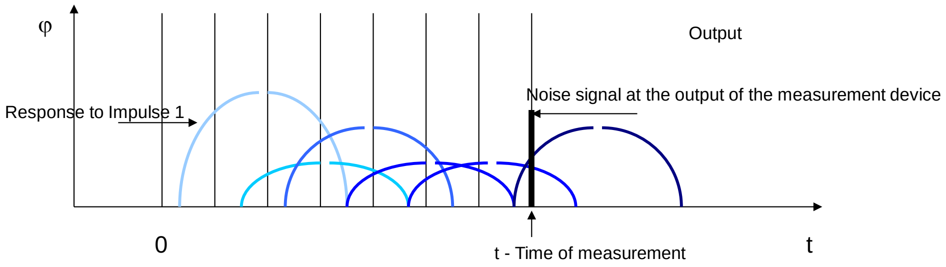


Figure4: Noise signal at the output of the measuring device at time t.

This is valid when the time intervals Δt\Delta tΔt are much smaller than any time constant in the system.

$$\varphi(t) = \sum_{t_i=-\infty}^{t} q_i \, P(t, t_i)(\,1)$$

$P(t, t_i)$ is the impulse response at time ttt to an input impulse with unit integral (Dirac impulse) that occurred at time t_i.

It follows that:

$$\varphi(t)^2 = (\sum_{t_i=-\infty}^{t} q_i \mathrm{P}(\mathrm{t}, t_i))^2 = \sum_{t_i, t_j=-\infty}^{t} q_i q_j \, P(t, t_i) P(t, t_j) \; (2)$$

Note the following: since we have a random signal at the input, we cannot calculate the actual value of $\varphi(t)$ Instead, we will calculate the following variance:

$$\langle \varphi(t)^2 \rangle_t$$

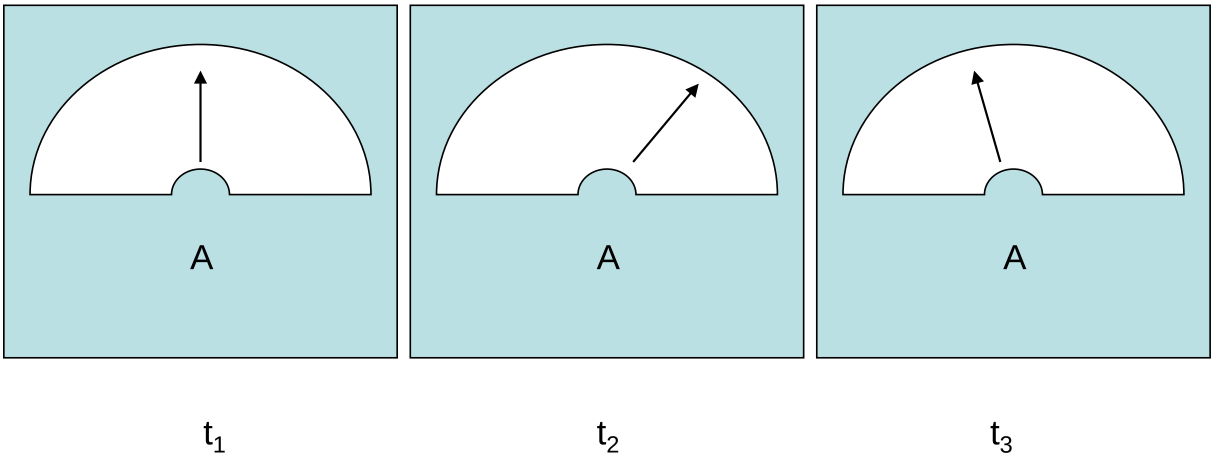


Figure5: One measuring device at different moments in time

It is difficult to calculate mean values in the time domain. Therefore, we make the following assumption:

It makes no difference whether we measure the output of a circuit (which does not change over time) at different moments in time (Figure 5) and calculate a statistical quantity from these samples, or whether we create many copies of this circuit (or imagine many copies) (Figure 6) and measure the output of each circuit at the same moment and then calculate statistics.

This is valid for **ergodic systems** that have a finite correlation time, i.e. finite time constants, such that noise signals at time ttt are independent of noise signals at time $t - \tau$ when τ is sufficiently large. For such systems, the autocorrelation function (ACF) is equal to zero

$$ACF(\tau) = \int_{-\infty}^{\infty} \varphi(t)\, \varphi(t+\tau)\, dt \, (3)$$

for τ greater than the time constants of the system.

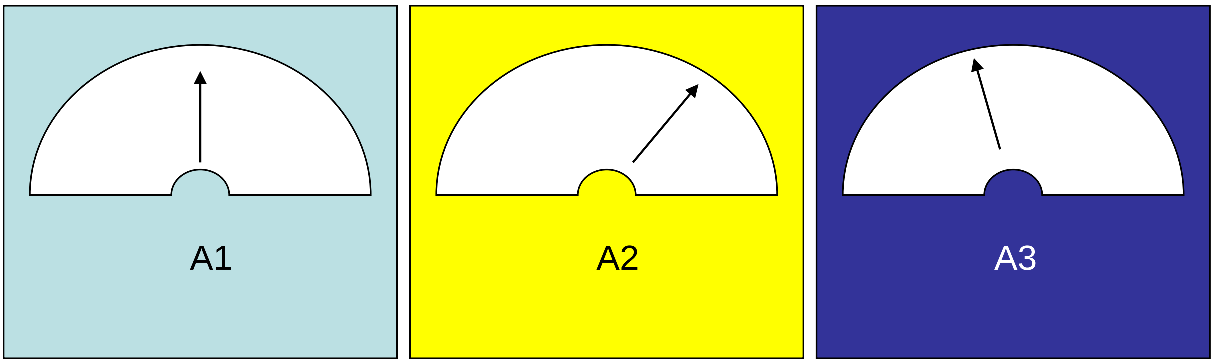


Figure6: Multiple copies of the measuring device (ensemble)

In the first case (Figure 5), we have **time statistics** (e.g. time average). In the second case (Figure 6), we have **sample statistics** or **ensemble statistics** (ensemble average).

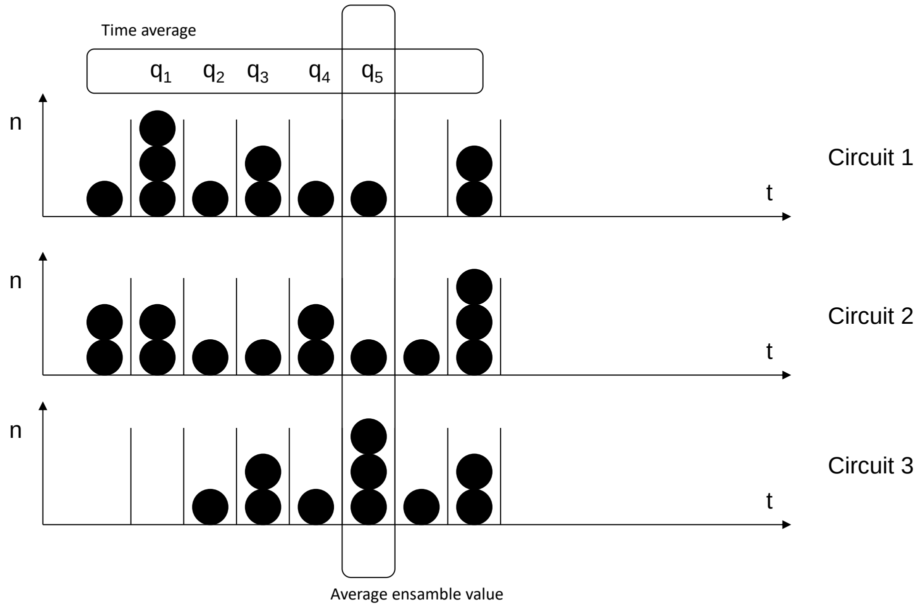


Figure7: Time average and ensemble average

The following holds:

$$\langle \varphi(t)^2 \rangle_t = \langle \varphi(t)^2 \rangle_e (4)$$

The time average is equal to the ensemble average (Figure 7).

If equation (2) is substituted into equation (4), we obtain:

$$\langle \varphi(t)^2 \rangle_e = \sum_{t_i,t_j=-\infty}^{t} \langle q_i q_j \rangle_e \, P(t,t_i) P(t,t_j) \, (5)$$

Note: the summation runs over time, while the averaging is performed over different copies of the circuit. For each product $q_i q_j$, there is an ensemble average.

The following applies:

Case 1: For different i and j $i \neq j$, the product of noise amplitudes $q_i q_j$ is equally likely to be positive or negative. Therefore, the mean value $\langle q_i q_j \rangle = 0$ for $i \neq j$.

Case 2: For i=j $q_i q_j = q_i^2$ which is positive. $\langle q_i^2 \rangle$ is non-zero. $\langle q_i^2 \rangle$ is the variance of q_i.

Accordingly,

$$\langle q_i q_j \rangle_e = \langle q^2 \rangle_e \, \delta_{ij} (6)$$

This simplifies equation (5) to

$$\langle \varphi(t)^2 \rangle_e = \sum_{t_i=-\infty}^{t} \langle q^2 \rangle_e \, (P(t, t_i))^2 \quad (7)$$

The summation in equation (7) can also be expressed as an integral.

The variance of the noise signal at the output at time t is:

$$\langle \varphi(t)^2 \rangle_e = \int_{-\infty}^{t} \frac{\langle q^2 \rangle_e}{\Delta t} \, (P(t, \tau))^2 \, d\tau \quad (8)$$

Δt is the length of the time intervals.

$P(t, \tau)$ is the impulse response at time ttt to an input impulse with unit integral (Dirac impulse) that occurred at time τ: $\delta(t - \tau)$.

The quantity q_i is the noise amplitude at the input during the interval Δt, with variance $\langle q_i^2 \rangle$.

Equation (8) is the formula for the variance of the noise signal in the time domain.

Example 1: Leakage current noise of a photodiode

We will now analyze the following circuit:

A photodiode (Figure 8) uses a PN junction (a diode) biased in reverse direction to detect light. The reverse bias voltage enlarges the depletion region, strengthens the electric field, and makes charge collection more efficient.

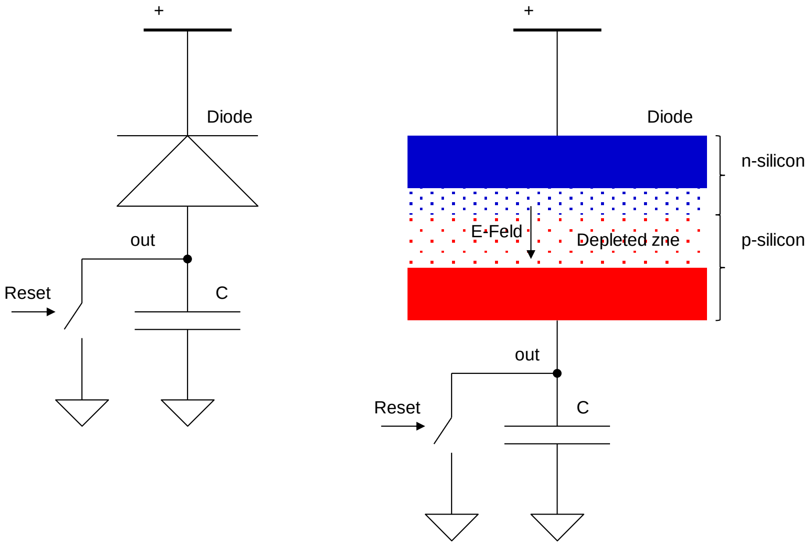


Figure8: Photodiode

When a photon is absorbed in silicon (Figure 9(a)), and if its energy is greater than 1.1 eV (the silicon band gap), an electron can be excited from the valence band into the conduction band (b). In this way, an electron–hole pair is generated.

If this occurs in the depletion region, where the electric field is strong, the electron is attracted toward the N-side (c). The hole is attracted toward the P-region; equivalently, an electron moves through the valence band and fills the hole (d). In this way, a photocurrent is generated.

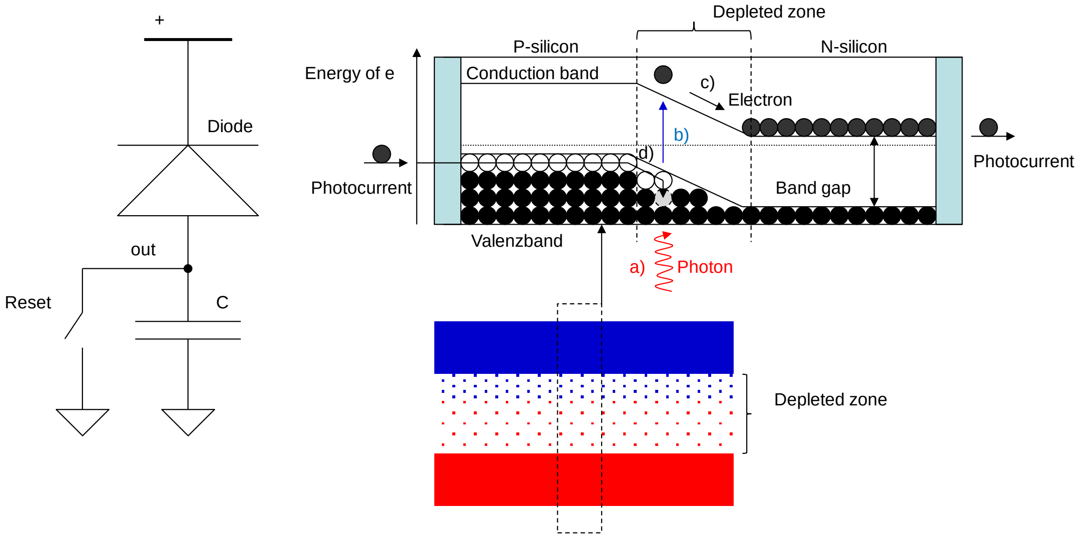


Figure9: Generation of photocurrent

A pixel sensor (Figure 10) consists of a matrix of such photodiodes. A pixel contains the photodiode, an amplifier AAA to measure the diode voltage, a reset switch (S1S_1S1), and a select switch (S2S_2S2). All pixels in a column share the readout line. All pixels in a row share the select and reset lines.

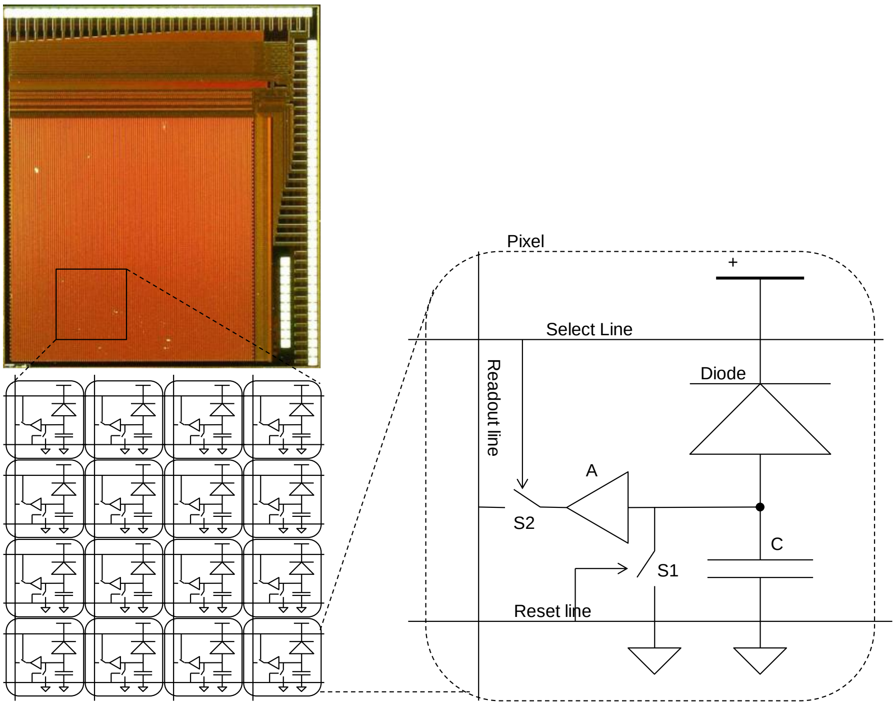


Figure10: Pixel matrix and block diagram of a pixel

The readout circuit can be modeled as a capacitor C (Figure 11), which is connected to the diode. The capacitor is discharged using a switch (a MOSFET) during the reset phase. The diode current (photocurrent i_{signal}) flows into the capacitor C for a certain integration time (exposure time). After the exposure time T_{exp} the voltage across the capacitor is measured.

The voltage increases when the light intensity is higher, since the number of absorbed photons and the photocurrent are larger.

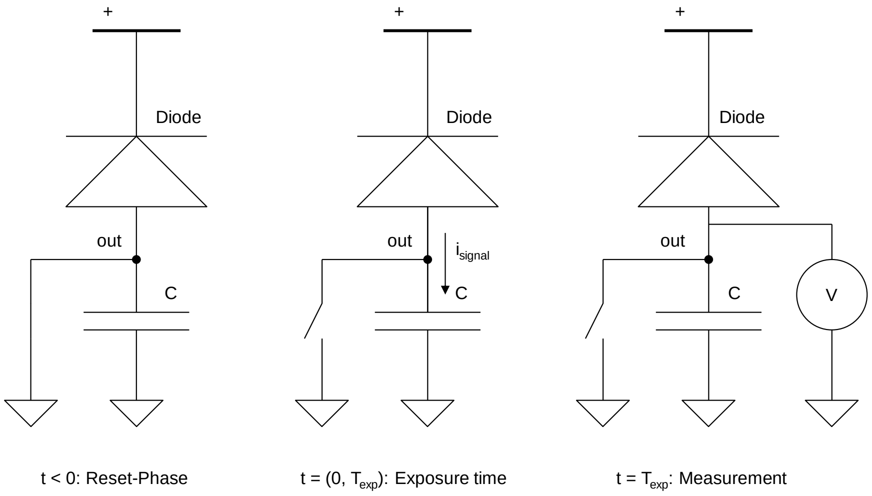


Figure11: A pixel circuit (photodiode) in three phases

In a diode that is biased in reverse direction, a current flows even in the absence of light (dark current). This is a so-called **temperature-induced leakage current**, caused by electron–hole pairs generated through thermal processes (Figure 12).

The mechanism can be visualized as follows: temperature leads to vibrations of the crystal lattice. The quantum of these vibrations is called a **phonon**. A phonon can transfer its energy to a valence electron and generate an electron–hole pair. In this way, the leakage current is produced.

This leakage current leads to an offset in the measurement. The offset would be unproblematic if it were constant: it could be measured without light and later subtracted from the measurement with light. Unfortunately, the leakage current is noisy, because thermal generation (absorption of phonons) is a stochastic process. Leakage current noise causes the offset to vary between different measurements (time statistics) or between different pixels (ensemble statistics). As a result, noise appears in the image.

We will now quantify this noise, i.e. calculate the variance of the voltage across the capacitor.

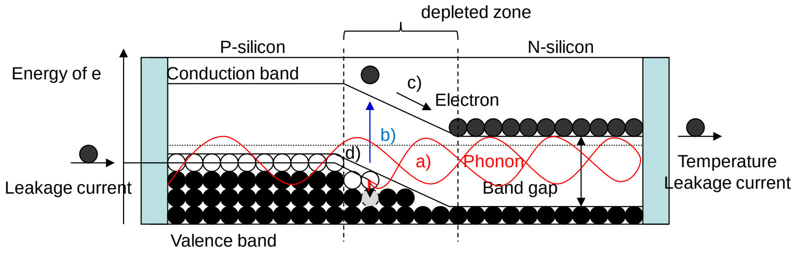


Figure12: Temperature leads to vibrations in the crystal lattice. The quantum of these vibrations is called a phonon. A phonon can also generate an electron–hole pair. In this way, a leakage current is produced.

We will use the formula for the noise signal in the time domain (8). In our system, the capacitor plays the role of the measuring device. The input signal is the leakage current.

For equation (8), we need $\langle q^2 \rangle$ and the impulse response.

Variance of the noise amplitude

How large is the variance of the noise amplitude $\langle q^2 \rangle$?

The unit of q is coulombs. Q_i is the charge generated in the diode during the time interval Δt (Figure 13).

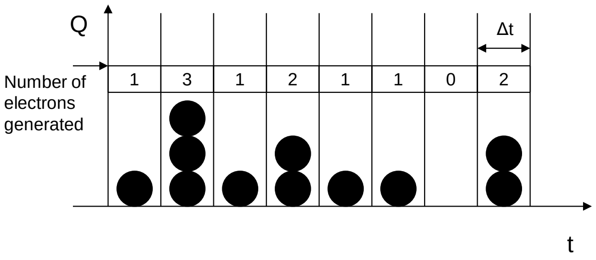


Figure13: The number of thermally generated electrons in Δt is a random variable

The charge is generated by the creation of electron–hole pairs. The individual generation events are independent. The number of electrons generated within a time interval is a random variable. For these reasons (independence of events and randomness in time), the number of electrons generated in Δt follows a Poisson distribution.

The Poisson distribution is defined by the following formula:

$$P_\lambda(x = n) = \frac{\lambda^n}{n!} e^{-\lambda} (9)$$

P is the probability that a random variable x takes the value n. The constant λ is the mean value of x. Figure 12 shows the Poisson distribution as a graph.

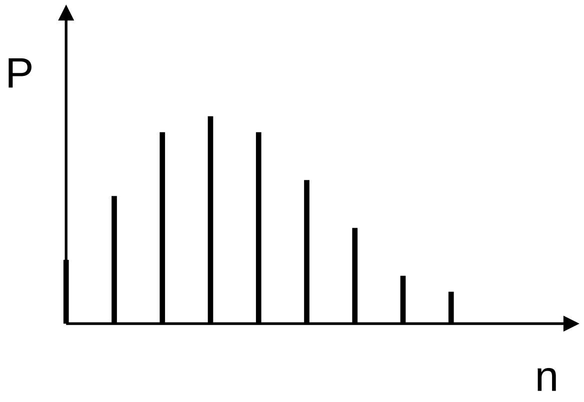


Figure14: Poisson distribution. P is the probability that a random variable x takes the value n. The constant λ is the mean value of x

The Poisson distribution is important for noise theory. It is the standard distribution for integer-valued random variables, just as the Gaussian distribution is the standard distribution for real-valued random variables.

The Poisson distribution has the property that the variance (the square of the standard deviation) is equal to the expected value (mean).

The Poisson distribution is a good model for events that occur independently of one another with a constant average rate within a fixed time interval or spatial region.

As an example, consider a PN diode biased in reverse direction. The leakage current is generated by the thermal generation of electron–hole pairs. The number of generated electrons is a random variable that follows a Poisson distribution.

Therefore, the following holds:

$$\langle q^2\rangle_e = \langle q^2\rangle_t = e^2\langle n^2\rangle_t = e^2\langle n\rangle_t = eI\Delta t \ (10)$$

Impulse Response

Let us now derive the impulse response $P(t = T_{\mathrm{exp}}, \tau)$.

$P(t, \tau)$ is the impulse response at time ttt to an input impulse with integral 1 that occurred at time τ.

If a charge impulse 1C × $\delta(t - \tau)$ passes into the capacitor during the time τ=(0,Texp)\tau = $\tau = (0, T_{\mathrm{exp}})$ we measure the capacitor voltage $1/C$ ×Coulomb at the moment T_{exp} (Figure 15 – B).

If a charge impulse 1C × $\delta(t - \tau)$ occurs earlier $\tau < 0$ (Figure 15 – A), we measure a voltage of 0, since the reset switch is closed for t<0. Similarly, for charge impulses 1C × $\delta(t - \tau)$ at $\tau > T_{\mathrm{exp}}$, we measure 0 voltage at $t = T_{\mathrm{exp}}$ (Figure 15 – C).

Therefore, the impulse response is

$$P(T_{\mathrm{exp}}, \tau) = \frac{1}{C} \text{for} \tau \in (0, T_{\mathrm{exp}}),$$

and otherwise

$$P(T_{\mathrm{exp}}, \tau) = 0$$

(Figure 15, top).

Note that the impulse response does not have units of Coulombs, since an impulse response is the response to a Dirac impulse with unit magnitude. Only when we multiply the impulse response by an amplitude q do we obtain an output voltage.

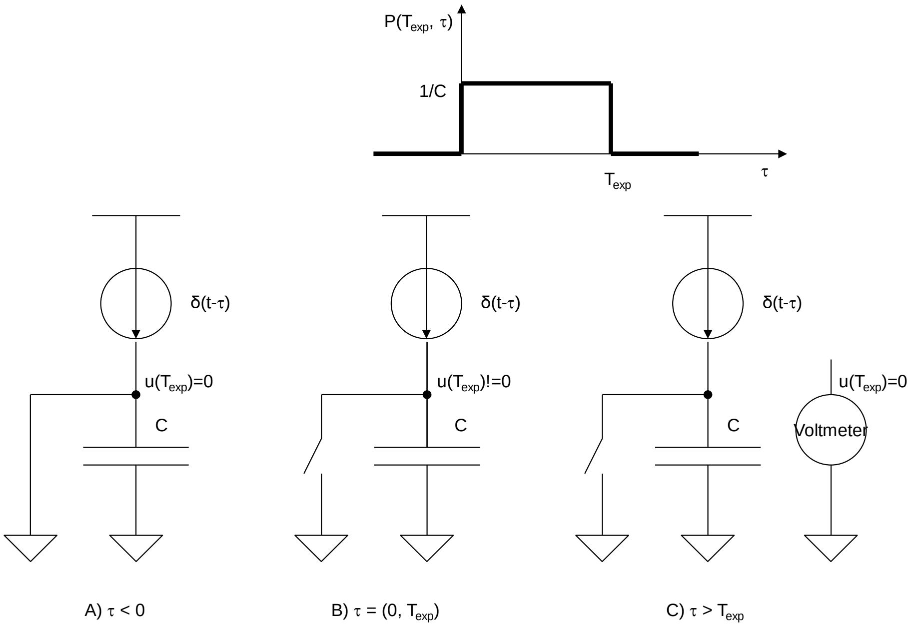


Figure15: Derivation of the impulse response $P(T_{exp}, \tau)$ Response to an impulse

Summary:

Variance of the noise signal:

$$\langle \phi(t)^2 \rangle_e = \int_{-\infty}^{t} \frac{\langle q^2 \rangle_e}{\Delta t} \, (P(t,\tau))^2 \, d\tau$$

Variance of the noise amplitude:

$$\langle q^2 \rangle_e = eI\Delta t$$

Impulse response:

$$P(T_{\mathrm{exp}},\tau) = \frac{1}{C}, \tau \in (0, T_{\mathrm{exp}})$$

$$P\left(T_{\mathrm{exp}},\tau\right) = 0, \tau < 0 \text{ or } \tau > T_{\mathrm{exp}}$$

If we insert the formulas for the variance of the noise amplitude (10) and the impulse response (11) into (8), we get:

$$\langle u_{\mathrm{out}}(T_{\mathrm{exp}})^2 \rangle = \int_{-\infty}^{T_{\mathrm{exp}}} eI \, (P(T_{\mathrm{exp}},\tau))^2 \, d\tau = \int_{0}^{T_{\mathrm{exp}}} \frac{eI}{C^2} \, d\tau = \frac{eIT_{\mathrm{exp}}}{C^2} \, (12)$$

The variance of the measured voltage at time T_{exp} is thus proportional to the charge accumulated on the capacitor by the leakage current I during the measurement time T_{exp} ($Q = IT_{\mathrm{exp}}$).

Discussion

Noise by itself tells us little about the quality of a measurement. Only when we know the signal amplitude can we determine whether the noise is large or small.

The signal at the photodiode is:

$$U_{\mathrm{sig}} = \frac{I_{\mathrm{sig}}T_{\mathrm{exp}}}{C} \, (13)$$

where I_{sig} is the photocurrent.

A useful quantity is the **signal-to-noise ratio (SNR)**. The SNR can be defined either as the ratio of signal power to noise power, or as the ratio of signal amplitude (e.g., in volts) to the standard deviation of the noise (also in volts). In English, the standard deviation is often referred to as the **root mean square (RMS)**.

We will calculate the SNR by dividing the signal voltage U_{sig} by the standard deviation (square root of the variance) of the noise:

$$\mathrm{SNR} = \frac{(I_{\mathrm{sig}} T_{\mathrm{exp}})/C}{\sqrt{(eIT_{\mathrm{exp}})/C^2}} = \frac{I_{\mathrm{sig}}\sqrt{T_{\mathrm{exp}}}}{\sqrt{eI}} \quad (14)$$

How can the SNR of the photodiode be improved?

1. By reducing the leakage current I, which is temperature-dependent.
2. By increasing the measurement time T_{exp}

Are smaller pixels less noisy?

Smaller pixels do have lower leakage current (and thus a lower RMS value of the noise signal), but they also produce less signal current. Since the denominator in (14) depends on the square root of the leakage current and the numerator on the signal current, smaller pixels actually have worse SNR, despite their lower noise level.

Noise in the Frequency Domain

The formula for noise in the time domain (8) is suitable for systems that change over time, such as circuits that contain switches. If a system remains unchanged for a longer time, it is often easier to use a formula in the frequency domain, because instead of the impulse response, one can use the **transfer function**. We will now derive the formula for the variance of the noise signal at the output in the frequency domain.

If the system is **stationary**, the following applies for the impulse response $P(t,\tau)$:

P(t,τ) is the impulse response at time t to a Dirac impulse δ(t−τ))

$$P(t,\tau) = P(t-\tau,0) \equiv P(t-\tau)(15)$$

Here, P is the response to the Dirac impulse δ(t).

Starting from equation (8):

$$\langle \phi(t)^2 \rangle_e = \int_{-\infty}^{t} \frac{\langle q^2 \rangle_e}{\Delta t} \left(P(t,\tau)\right)^2 d\tau$$

If we insert (15) into (8), we get:

$$\langle \phi(t)^2 \rangle_e = \int_{-\infty}^{t} \frac{\langle q^2 \rangle_e}{\Delta t} \left(P(t-\tau)\right)^2 d\tau = \int_{0}^{\infty} \frac{\langle q^2 \rangle_e}{\Delta t} \left(P(u)\right)^2 du$$
$$= \int_{-\infty}^{\infty} \frac{\langle q^2 \rangle_e}{\Delta t} \left(P(u)\right)^2 du \, (16)$$

In the last step, we used P(u)=0 for u<0.

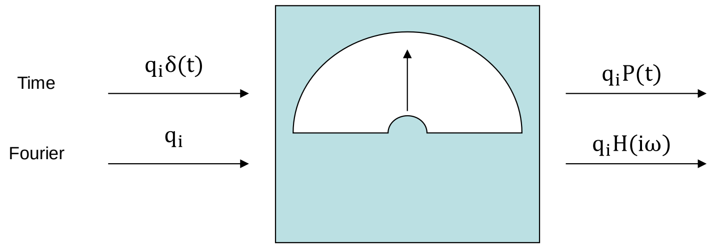
According to Parseval’s theorem (see Wikipedia):

$$\int_{-\infty}^{\infty} \frac{\langle q^2 \rangle_e}{\Delta t} \left(P(t)\right)^2 \, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\langle q^2 \rangle_e}{\Delta t} \, | \, P(i\omega) \, |^2 \, d\omega \, (17)$$

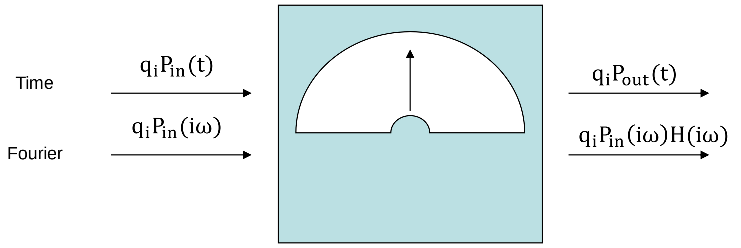
Here, P(iω) is the **Fourier transform** of the impulse response. It can be shown that P(iω) is equal to the **transfer function** H(iω):

$$P(i\omega) = H(i\omega) = \frac{V_{\text{out}}(i\omega)}{V_{\text{in}}(i\omega)} (18)$$

This can be derived by solving the linear circuit using complex impedances.



White noise



Frequency-dependent noise

Figure16: Measuring device with transfer function H(iω)

Let us consider equation (17) more generally: Our measurement device receives **noise impulses** at its input and produces **noise impulse responses** at its output (Figure 16).

We assumed that the input impulses (noise impulses) have the form of Dirac impulses, each multiplied by independent amplitudes q_i (Figure 16, top).

This assumption is valid for **white noise**. Noise is called white if the Fourier transform of the input impulses (Dirac impulses) is a constant function. The input noise thus contains all frequencies, analogous to white light containing all wavelengths.

There are also types of noise where the noise impulses last longer than Δt and have a time profile $q_i \times P_{\text{in}}(t)$. In this case, instead of using the impulse response, one must use the response to $P_{\text{in}}(t)$ denoted P_{out}, when deriving the output noise signal (Figure 16, bottom).

The Fourier transform $P_{\text{out}}(i\omega)$ is then the **product** of the Fourier transform of the noise impulse $P_{\text{in}}(i\omega)$ and the Fourier transform of the impulse response $H(i\omega)$. If the impulses $P_{\text{in}}(t)$ are short (Dirac impulses), then $P(i\omega) = 1$

The variance of the noise signal at the output of a **stationary system** is:

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\langle q^2\rangle_e}{\Delta t}\;|\,P_{\text{in}}(i\omega)\,|^2\;\;|\,H(i\omega)\,|^2\;\;d\omega\;(19)$$

We define here the **spectral power density** of the noise source as:

$$S(f) = 2\frac{\langle q^2\rangle}{\Delta t}\;|\,P_{\text{in}}(i\omega)\,|^2\;(20)$$

Substituting (20) into (19), we get:

$$\langle \phi^2 \rangle = \frac{1}{4\pi}\int_{-\infty}^{\infty} S(f)\ \mid H(i\omega) \mid^2\ d\omega = \frac{1}{2}\int_{-\infty}^{\infty} S(f)\ \mid H(i\omega) \mid^2\ df$$
$$= \int_{0}^{\infty} S(f)\ \mid H(i\omega) \mid^2\ df\ (21)$$

This is the formula for the variance of the noise signal in the **frequency domain**.

This variance is equal to the **mean square (MS) value** in the time domain. Another name for this quantity is the **noise power**.

Here, $H(i\omega)$ is the **transfer function** of the circuit, and S(f) is the **spectral power density**, defined by (20).

As mentioned, if we assume that the input impulses are very short and that the approximation $P_{\text{in}}(i\omega) = 1$ holds, we are dealing with **white noise**.

Leakage current noise is white noise when a relatively large number of electrons are generated within a time interval that is shorter than any system time constant.

There are also types of noise where $P_{\text{in}}(i\omega) \neq 1$.

An example is **1/f noise** (also called **flicker noise**). For 1/f noise, we have

$$P_{\text{in}}(i\omega) \sim \frac{1}{\omega}.$$

How do we measure the noise signal?

In the derivation, we used the **ensemble average** (mean over many realizations). In reality, we do not have multiple copies of the circuit. However, we **can repeat the measurement on the same system**. If the time between measurements is much longer than the system's time constants, the measurements are uncorrelated. In that case, the mean based on these repeated measurements is equivalent to the ensemble average.

This means we can use a **digital oscilloscope** with an **RMS measurement function** to measure $\langle \phi^2 \rangle$.This function continuously measures ϕ(t) and calculates the time-averaged square. Here, ϕ(t) is the "small signal," so the DC component must be subtracted. It is important that the measurement is long enough to obtain a reliable result.

We can also take **multiple samples of ϕ**, e.g., with the oscilloscope at different times t1, t2,…, read the amplitudes ϕ(t1), ϕ(t2),… and then calculate the variance. In both cases, we obtain the same result if the number of samples and the measurement time are sufficiently large.

Leakage Current Noise in the Frequency Domain

Example: How large is the spectral power density S(f) in the case of leakage current noise?

We defined the spectral power density S(f) as follows (20):

$$S(f) = 2\frac{\langle q^2\rangle}{\Delta t}\ \mid P_{\mathrm{in}}(i\omega)\mid^2$$

According to formula (10):

$$\langle q^2\rangle_e = eI\Delta t(22)$$

For white noise, $P_{\mathrm{in}}(i\omega) = 1$.

Substituting (22) into (20), we obtain the spectral power density for leakage current noise:

$$S(f) = 2eI(23)$$

Another name for this type of noise is **shot noise**. This name comes from the analogy between the flow of electrons and the pellets (“shots”) forming in a free-falling tower (see shot tower).

Shot noise (also called **Poisson shot noise** or **Schottky noise**) is a form of white noise in optics and electronics that can be modeled as a **Poisson process**.

Thermal Noise in the Frequency Domain

How large is the spectral power density for the **thermal noise** of a resistor?

In a resistor, there is no generation of electrons, so it is initially unclear whether we can use formula (23).

If we substitute the **average drift current** flowing through a resistor (U/R) into formula (23), we would obtain a spectral power density that is far too large.

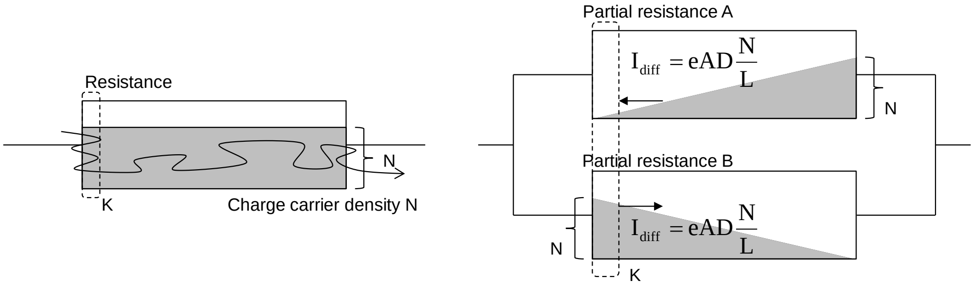


Figure17: Thermal noise in a resistor

However, we can imagine the following: The electrons in a resistor move not only because of the applied field (drift current) but also due to their **kinetic (thermal) energy**. This **thermal current** leads to a noise amplitude q. This amplitude is the **amount of charge q** within a contour K (Figure 17).

How large are the **variance of q** and the thermal current?

We can model the resistor as a **parallel connection of two sub-resistors, A and B** (Figure 17, right). The two sub-resistors have the same lengths and linearly increasing and decreasing carrier densities, from 0 to N (A) and from N to 0 (B). Here, N is the carrier density of the original resistor. The total resistance of the parallel connection is also R, so this decomposition is allowed.

The **density gradient** leads to the following **diffusion current**:

$$I_{\text{diff}} = eAD\frac{N}{L} \quad (24)$$

where A and L are the cross-section and length of the resistor, D is the diffusion constant, and e is the elementary charge.

Resistor A transports, within a time Δt, an amount of charge q_A into the contour K, while resistor B transports an amount q_B out of contour K. The total charge in the contour is therefore:

$$q = q_A + q_B$$

The charges q_A and q_B are **random variables** with the same mean:

$$\langle q_A \rangle_e = \langle q_B \rangle_e = I_{\text{diff}}\,\Delta t$$

For the variances, we have:

$$\langle q^2\rangle_e = \langle q_A^2\rangle_e + \langle q_B^2\rangle_e (25)$$

For each variance, equation (22) applies:

$$\langle q_A^2\rangle_e = \langle q_B^2\rangle_e = eI_{\text{diff}}\,\Delta t (26)$$

If we substitute (25) and (26) into (20), we obtain the **spectral power density**:

$$S(f) = 2\frac{\langle q^2\rangle}{\Delta t}\;|\,P_{\text{in}}(i\omega)\,|^2 = 2\frac{\langle q_A^2\rangle + \langle q_B^2\rangle}{\Delta t} = 4eI_{\text{diff}} (27)$$

If we substitute (24) into (27), we get:

$$S_R(f) = 2S_{R1}(f) = 4e^2AD\frac{N}{L} (28)$$

Let us now express the **spectral power density** (28) as a function of R:

The resistor is described by the formula:

$$\frac{1}{R} = e\,\mu\,N\,\frac{A}{L} (29)$$

where μ\muμ is the **mobility of the charge carriers**.

The **Einstein relation** connects mobility and the diffusion constant:

$$\frac{D}{\mu} = \frac{kT}{e} (30)$$

Here, k is the Boltzmann constant and T is the temperature.

If we substitute (30) and (29) into (28), we obtain the **spectral power density** of the thermal current noise in a resistor:

$$S_{I_R}(f) = \frac{4kT}{R} (31)$$

The noise of a resistor can thus be represented by a **current source** with spectral power density $4kT/R$. The unit of S_{I_R} is A^2/Hz.

This thermal noise is also called **Nyquist noise**, **Johnson noise**, or **Johnson–Nyquist noise**.

We can transform the **current source with a parallel resistor** into a **voltage source with a series resistor** (Figure 18). For the voltage and current of the respective sources, the relation is:

$$U = I \times R$$

The **spectral power density** of the voltage source is then:

$$S_V = R^2 S_I$$

This gives:

$$S_{V_R}(f) = 4kTR(32)$$

The noise of a resistor can thus be represented by a **voltage source** with spectral power density $4kTR$. The unit of S_{V_R} is V^2/Hz.

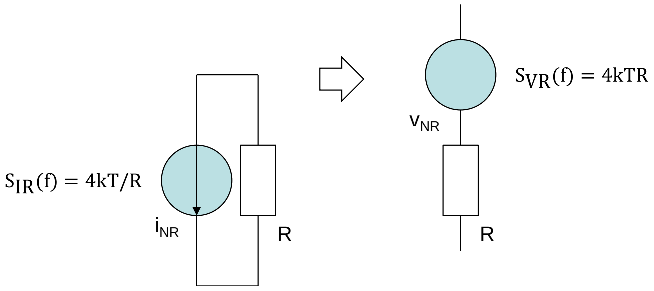


Figure18: Noise of a resistor can be modeled either by a current source or by a voltage source

Example: Reset Noise

A photosensor with capacitance C must be discharged (reset) before each measurement using a switch (resistor R) (Figure 19). The resistor R generates **voltage noise** on the capacitor C. When the switch is opened, a **voltage error** remains on the capacitor. The **variance** of this error can be calculated using formula (8).

However, if the switch has been closed for a long time (significantly longer than the time constant RC), the **stationary formula** (19)—the formula for the variance of the noise signal, or noise power in the frequency domain—can also be used:

$$\langle u^2 \rangle = \int_0^{\infty} S_{V_R}(f) \ | H(i\omega) |^2 \ df \ (33)$$

The **transfer function** of the RC circuit is:

$$H(i\omega) = \frac{V_{\text{out}}(i\omega)}{V_{nR}} = \frac{1}{1 + i\omega RC}(34)$$

The **spectral power density** is (32):

$$S_{V_R} = 4kTR(35)$$

Substituting (34) and (35) into (33), we get:

$$\langle u^2 \rangle = \int_0^\infty 4kTR \mid \frac{1}{1+i\omega RC} \mid^2 df = \frac{4kTR}{2\pi} [\frac{1}{RC} \arctan(\omega RC)]_0^\infty = \frac{kT}{C} (36)$$

The variance of the noise signal **does not depend on R**! Larger resistances generate more voltage noise, but they also produce longer time constants, which reduce the integral, so the two effects **cancel each other out**.

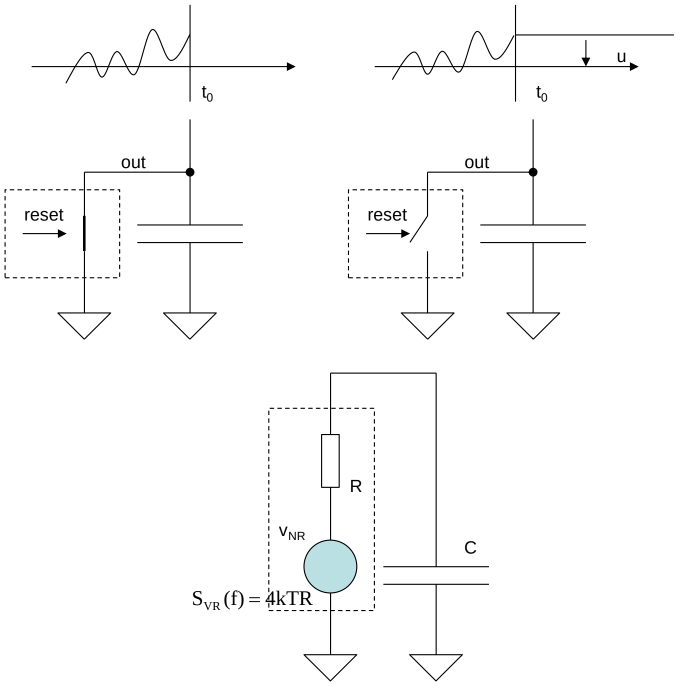


Figure19: Calculation of reset noise

Discussion

In a pixel sensor, we have **two sources of noise**:

1. Leakage current:

$$\langle u^2 \rangle = \frac{eIT_{\mathrm{exp}}}{C^2} (37)$$

where T_{exp} is the **exposure time**, C is the sensor capacitance, I is the leakage current, and e is the elementary charge.

2. Reset switch:

$$\langle u^2 \rangle = \frac{kT}{C} (38)$$

where k is the **Boltzmann constant** and T is the temperature.