Lecture 6

This lecture covers following themes:

Basics of frequency analysis, Bode plot, step response, high-pass and low-pass filter

Frequency analysis and step response of the inverting voltage amplifier

Charge-sensitive amplifier

Method for fast estimation of coefficients in characteristic polynomial

Basics of frequency analysis

The theme of this lecture are frequency-dependent circuits.

As a first example, we consider the simplest low-pass filter, which consists of a resistor and a capacitor, Fig 1. The output of the circuit is taken across the capacitor.

We can derive the complex transfer function as the ratio between the output and input voltages: $u_{out}(s)/u_{in}(s)$

Here, $u_{out}(s)$ is the Laplace transform of the output voltage $u_{out}(t)$. $u_{in}(s)$ is the Laplace transform of the input voltage $u_{in}(t)$.

The transfer function can be derived by solving the circuit using the node-voltage method, Kirchhoff's rules, or other circuit analysis techniques. To do this, the formulas for complex impedance must be applied. These formulas can, in turn, be derived from the underlying differential equations describing the currents and voltages in the circuit.

Capacitor: $i_c = C\ D\ u_c$

Inductor: $u_l = L\ D\ i_l$

Resistance: $u_r = R\ i_r$

Symbol D is the differential operator $D \equiv d/dt$.

If the differential operator D is replaced by the complex frequency s, we obtain the following Laplace transformed equations:

Capacitor: $i_c = s\ C\ u_c$

Inductor: $u_l = s\ L\ i_l$

Resistance: $u_r = R\ i_r$

And the corresponding impedances:

Impedance of capacitor $Z_c = 1/sC$ (1)

Impedance of inductivity $Z_l = sL$ (2)

Impedance of resistance $Z_R = R$ (3)

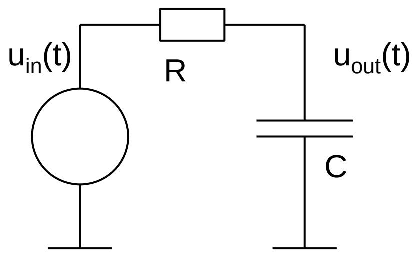


Fig 1: Low-pass filter

The simplest way to find the output voltage of the low-pass circuit is by using the formula for the voltage divider:

$$u_{out} = u_{in} \frac{Z_C}{Z_C + Z_R} \qquad (4)$$

When we substitute the formulas for impedances (2) – (4) into (5) we obtain:

$$u_{out}(s) = u_{in}(s) \frac{1}{1 + sRC} \qquad (5)$$

We can also write the differential operator D instead of frequency s in (6) and multiply both sides with the factor in the denominator. Then we get the linear differential equation:

$$(RCD + 1)u_{out}(t) = RC \frac{du_{out}(t)}{dt} + u_{out}(t) = u_{in}(t) \qquad (6)$$

U_{in} can be any function of time. A step function (or Heaviside function) is particularly useful for analysis. The circuit's **response to a step input** is called the **step response**.

The impulse response of a circuit is its response to a Dirac impulse (with area 1) applied at the input. It should not be confused with the step response, which is the response to a step function at the input. The impulse response can be obtained from the step response by taking its time derivative, while conversely, the step response is the integral of the impulse response. Alternatively, the impulse response can be derived directly from the transfer function V_{out} (s) / V_{in} (s) using the inverse Laplace or Fourier transform, since the Laplace or Fourier transform of a Dirac impulse is 1.

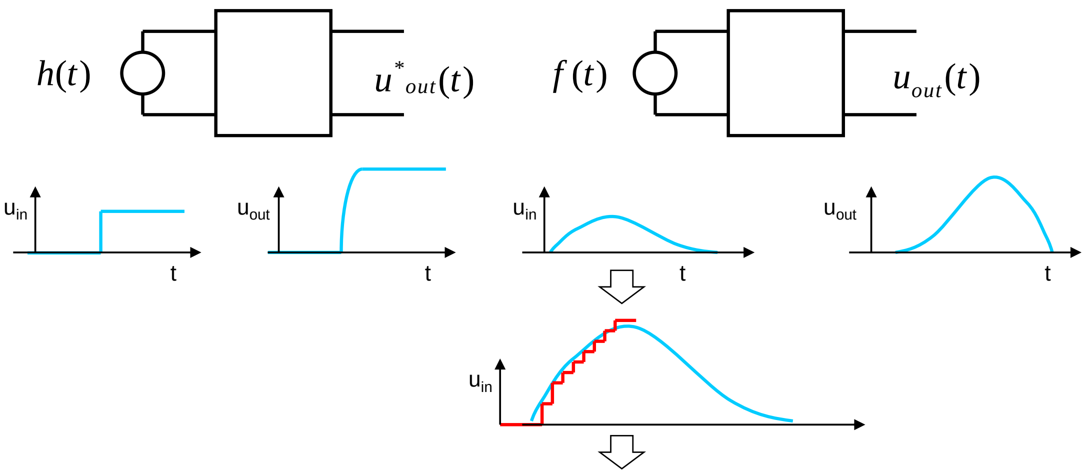
Why is the Step Response Important?

The step response can be derived intuitively.

It shows how the circuit reacts when the input suddenly jumps from 0 to a constant value.

From the step response $u^*_{out}(t)$, we can determine the response to any time-dependent function f(t) by convolving the step response with f'(t)

In this course we also analyse switched circuits, and their input signals often have the shape of a step function.



$$f(t) \approx \sum_{t=0}^{\infty} f'(t_i)\Delta t \cdot h(t-t_i)$$

$$u_{out}(t) = \sum_{t=0}^{\infty} f'(t_i)\Delta t \cdot u^*_{out}(t-t_i) = \int_0^{\infty} f'(\tau)u^*(t-\tau)d\tau$$

Fig 2: Response to a function f(t)

Calculation of the step response

Let us now calculate the step response of the low-pass circuit from Fig 1.

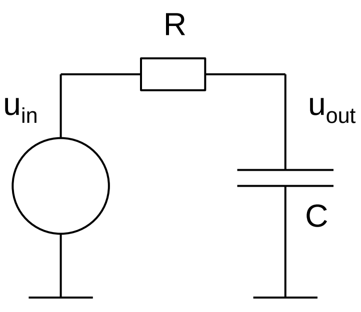


Fig 3: Low-pass filter

As mentioned above, we can derive the linear differential equation from the complex transfer function by replacing the complex frequency *s* with the differential operator D. This gives:

$(\mathrm{RCD} + 1)\mathrm{u}_{\mathrm{out}}(\mathrm{t}) = \mathrm{h}(\mathrm{t})$ (7)

H(t) is the step function with amplitude 1 , also called the Heaviside function.

We will not solve this equation completely. Instead, we will discuss the behavior of the circuit on two time scales:

1. **Short time scale** – shortly after the input voltage is "switched on."
2. **Long time scale** – after a long time, when the voltages have practically stopped changing.

For the **long time scale**, the terms containing time derivatives can be neglected because the voltages are no longer changing. The equation (8) then reduces to:

$$u_{out}(\infty) = 1$$

For short time scale (t << time constant RC), the terms without derivative can be neglected. The equation (8) reduces to:

$$\mathrm{RC}\ \mathrm{D}u_{out}(t) = h(t)$$

or

$$u_{out}(t) = \frac{1}{\mathrm{RC}}\frac{h(t)}{\mathrm{D}} = \frac{1}{\mathrm{RC}}\int h(t)dt = \frac{t}{\mathrm{RC}}$$

This is illustrated in Fig 4.

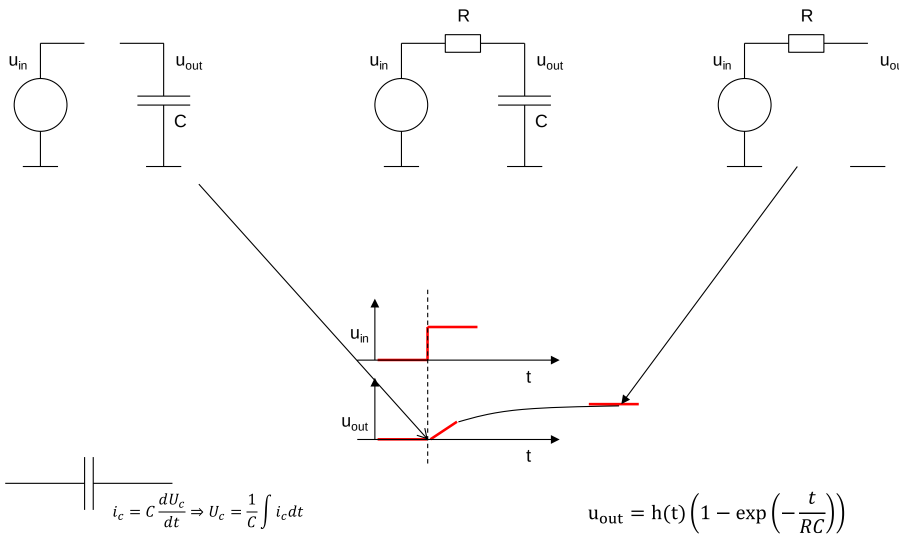


Fig 4: Solutions of $u_{out}(t)$ for short and long time scales

For intermediate times, we must solve the differential equation. Its solution has the general form:

$$A = C1e^{\lambda t} + A0$$

Here, λ is determined from the characteristic equation:

$$\mathrm{RC}\lambda + 1 = 0$$

It follows

$$\lambda = -\frac{1}{RC}$$

The product RC is called the time constant.

By choosing the constants C1 and A0 to satisfy the boundary conditions $u_{out}(0) = 0$ and $u_{out}(\infty) = 1$, we obtain the step response:

$$u_{out}(t) = h(t)(1 - e^{-\frac{t}{\tau}}) \qquad (8)$$

After approximately three time constants RC $u_{out}(t)$ reaches approximately 95% its maximum voltage.

Analogy of low-pass filter

Fig 5 illustrates an analogy for understanding the behavior of a low-pass filter:

Water level corresponds to voltage. Water flow corresponds to electric current. Just as the water level in a tank cannot rise instantaneously because the flow is limited, a capacitor cannot charge arbitrarily quickly because the current into it is limited by the circuit.

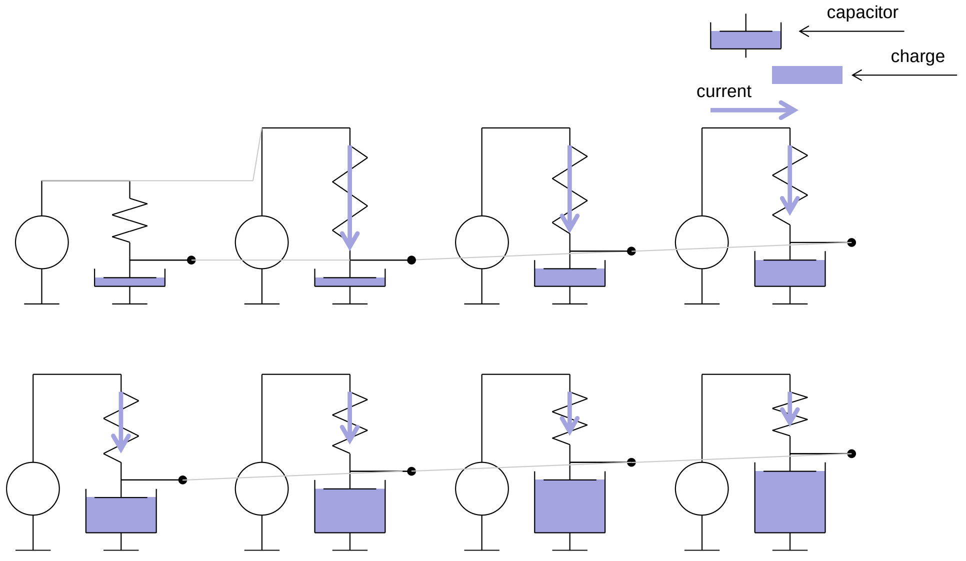


Fig 5: Low-pass filter analogy

Fig 6 shows the pulse time-response of the low-pass filter (left) and its Bode plot (right)

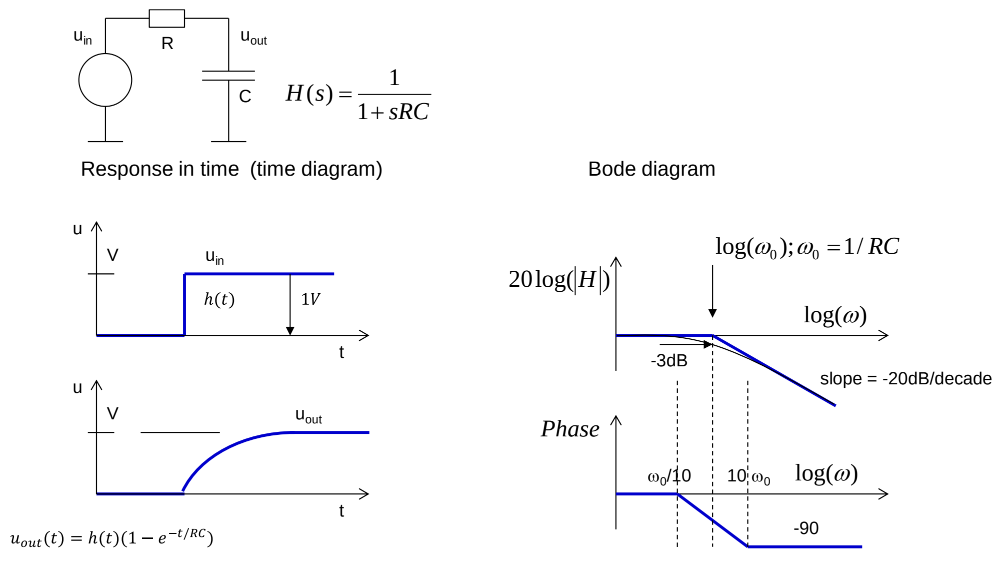


Fig 6: Low-pass filter – time dependent response (left) and Bode plot (right)

Bode plot

In a **Bode plot**, the **amplitude diagram** shows, 20 log of the absolute value of function $u_{out}(i\omega)/u_{in}(i\omega)$ is plotted against log(ω). In the phase plot, the phase of $u_{out}(i\omega)/u_{in}(i\omega)$ is plotted as a function of log(ω).

Bode plots of transfer functions with real zeros and poles are particularly simple to interpret. A pole is a solution of the polynomial in the denominator, and a zero is a solution of the polynomial in the numerator.

For a low-pass filter, the Bode plot behaves as follows:

For small frequencies the amplitude is 0 dB, meaning $u_{out} = u_{in}$.

At the pole frequency $\omega_0 = 1/RC$ it holds

$$|u_{out}| = \frac{|u_{in}|}{\sqrt{2}}$$

or 20 log(|H|) = -3dB.

For larger frequencies than pole it holds:

$|u_{out}| \sim |u_{in}|/(RC\ \omega)$

or

$20\log(|u_{out}/u_{in}|) \sim -20\log(RC) - 20\log(\omega)$.

This corresponds to a straight line with a slope of -20dB/decade which intersects the x-axis at the pole frequency 1/RC. The solid lines shown in the plot are the asymptotes.

The phase changes from 0° to -90° over a frequency range of approximately 0.1× to 10× the pole frequency (spanning about two decades).

Intuitive method to guess the step response

How can we intuitively predict the step response without solving the differential equations?

Long time scale:

For long times after the step is applied, the **time derivatives are zero**. Therefore:

For a **capacitor**, $i_C = 0$

For an **inductor**, $u_L=0$Applying this to the **low-pass filter**, we can remove the capacitors and short-circuit any inductors. This gives the long-time output:

$u_{out}(\infty)= 1$.

Short time scale:

Immediately after the step, the derivative terms dominate because RC d/dt is much larger than the DC terms. For a capacitor, this means $Z_c << R$. In a series connection of R and C, the resistor can be neglected, and the capacitor behaves like a short circuit. This gives the initial output:

$u_{out}(0) = 0$.

Using these **boundary conditions**, we can **intuitively sketch the step response** of the low-pass filter.

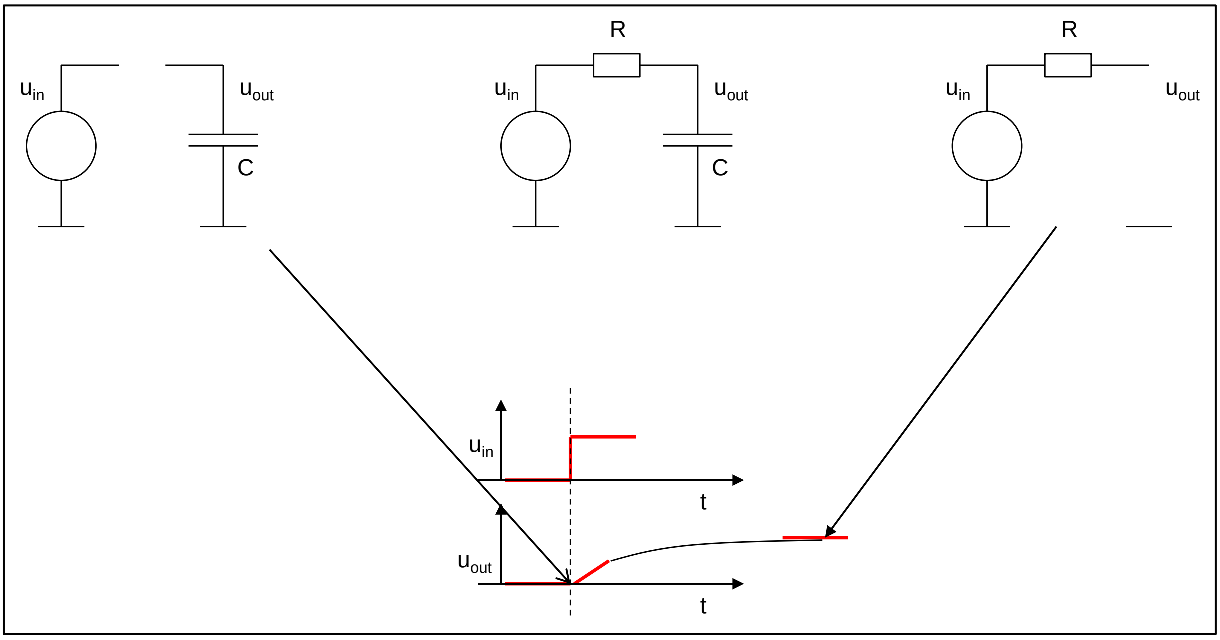


Fig 7: Quick estimation of step response

High-pass filter

Fig 8 shows high-pass filter, and its step response and Bode plot.

The transfer function of the high-pass filter is

$$u_{out}(s) = u_{in}(s)\frac{sRC}{1+sRC} \qquad (9)$$

And the step response

$$u_{out}(t) = h(t)e^{-\frac{t}{\tau}} \qquad (10)$$

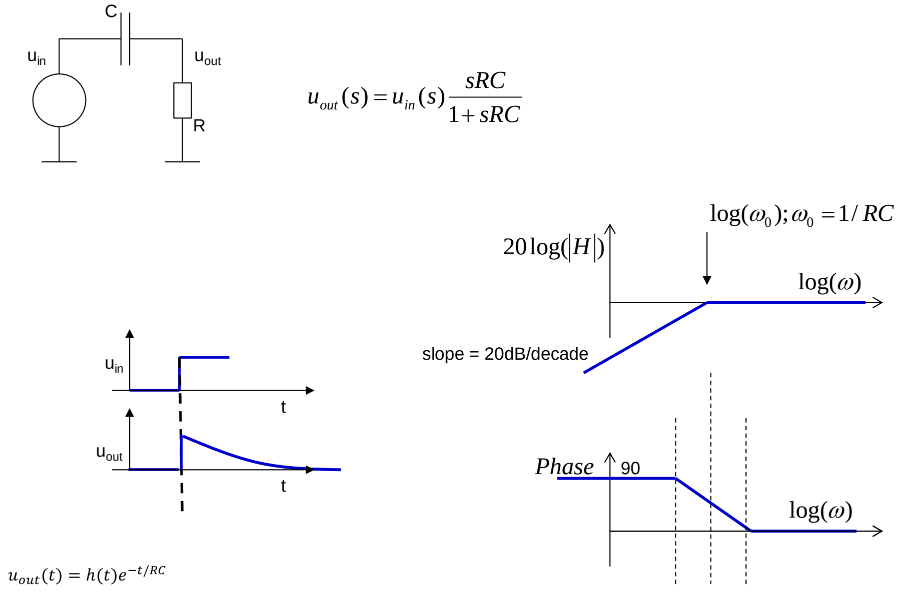


Fig 8: High-pass filter – time dependent response (left) and Bode plot (right)

Inverting voltage amplifier as a filter

The **voltage amplifier** is an important circuit, which we analyzed in the previous lecture. In that analysis, we **neglected the feedback resistor** R_{fb} and the **output resistor** R_{out} during the small-signal analysis. As a result, the formulas we derived previously were **frequency-independent**.

Figure 9 shows the **small-signal schematic** of the inverting voltage amplifier.

- C_o represents the **output capacitance**, which may be present when external circuits are connected to the amplifier.
- The **input signal** is a **step function**: $v_s(t) = h(t)$.

Simplified calculation

Let us now derive the output signal in a simplified way.

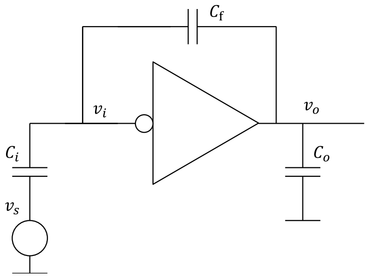


Fig 9: Inverting voltage amplifier

Let us assume that βA_{ol} is for every frequency large (Fig 10). βA_{ol} is in our case frequency dependent.

Case 1: R_f is neglected

For the moment let us neglect R_f.

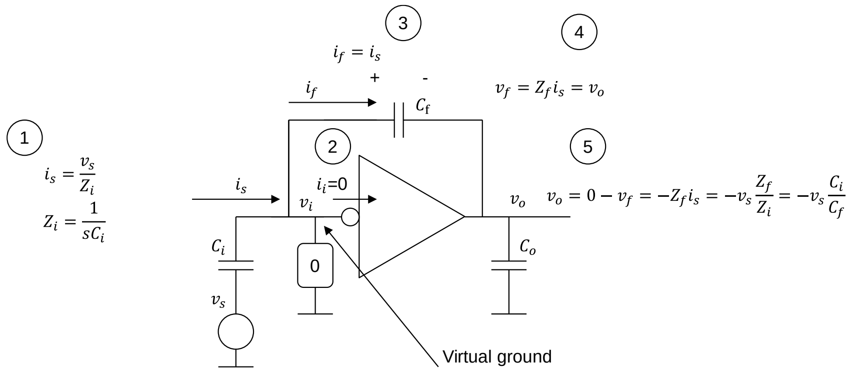


Fig 10: Fast calculation of amplification by assuming βA_{ol} >> 1 for every frequency - virtual ground

In this case (βA_{ol} >>1) the point at the input of the amplifier (v_i) behaves as a virtual ground, meaning the AC voltage at this point is zero.

We first calculate the **input current** $i_s(t)$, as shown in Fig 10.

$i_s = v_s/Z_i$

where Z_i is the impedance of the capacitor C_i. ($Z_i = 1/sC_i$)

The current through the feedback branch is: $i_f = i_s - i_i$.

Since $v_i = 0$ (virtual ground), $i_i = 0$, so

$i_f = i_s$.

The voltage across the feedback capacitor C_f is:

$v_f = Z_f \times i_f = Z_f \times i_s$.

The output voltage is then:

$$v_{out} = v_i - v_f = -v_f = -Z_f\, i_s = -v_s \frac{Z_f}{Z_i} \quad (11)$$

By substituting the capacitor impedances in (12), we get:

$$v_{out} = -v_s \frac{C_i}{C_f} \quad (12)$$

This is the same voltage gain formula we derived in Lecture 5

Note: This simplified analysis does not allow us to check whether the condition $\beta A \gg 1$ is still valid at higher frequencies.

Case 2: R_f is not neglected

How does the formula (12) change when we take R_f into account?

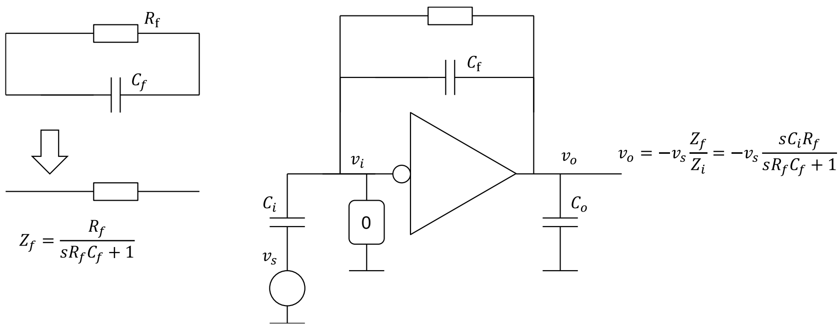


Fig 11: Influence of R_f to the transfer function

The formula (12) is remains valid, but the feedback impedance Z_f now has a different form (Fig 11):

$$Z_f = \frac{R_f}{1 + sR_fC_f} \quad (13)$$

This result is obtained by calculating the **parallel combination** of R_f and $Z_{cf} = 1/sC_f$

$$Z_f = \frac{Z_{cf}R_f}{Z_{cf} + R_f}$$

By substituting this Z_{cf} into (12), we obtain the **frequency-dependent transfer function** of the amplifier:

$$v_{out} = -v_s \frac{Z_f}{Z_i} = -\frac{sR_fC_i}{1+sR_fC_f} \quad (14)$$

How does the circuit respond to a step input h(t)?

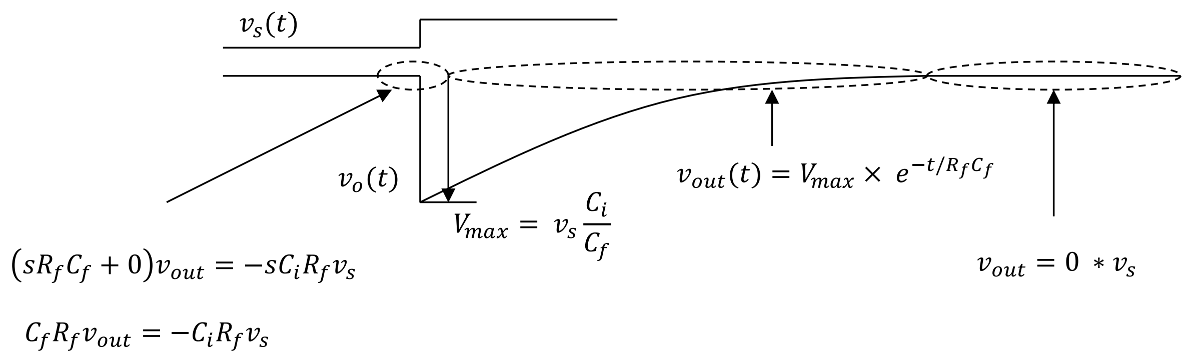


Fig 12: Solutions in different time intervals.

Short time scale: Only the terms containing s are significant. In this case, we have:

$$v_{out} = -\frac{C_i}{C_f}$$

It is the same formula, as (12) that derived without R_f.

Long time scale: For s=0s = 0s=0, substituting into the transfer function gives: $v_{out} = 0$.

Intermediate times: The full **time-dependent solution** requires solving the characteristic equation. The output voltage is:

$$v_o = -\frac{C_i}{C_f} e^{-\frac{t}{\tau}} \quad (15)$$

with the time constant $\tau = R_fC_f$.

This is illustrated in Fig 12.

The amplifier therefore behaves as a **high-pass filter**. Compare equation (15) with (10) for a low-pass filter. Since useful signals often **do not have a DC component**, an amplifier with high-pass characteristics is well suited. As mentioned earlier, R_f provides **DC feedback**, which is necessary to define the amplifier's **operating point**.

Exact calculation

We now perform a **more accurate calculation** by taking into account the **finite open-loop gain** -A and the output resistance R_{out}.

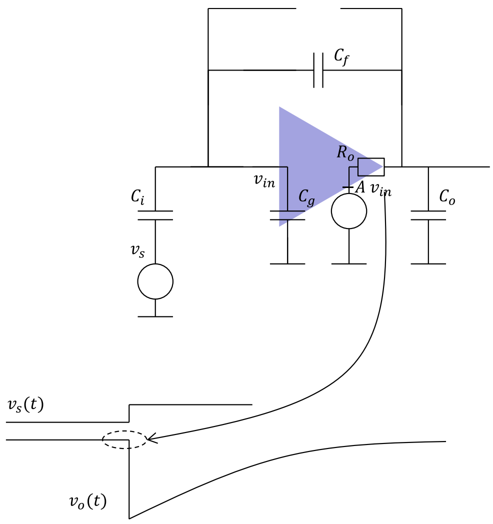


Fig 13: Calculation of step response for short time interval, R_f can be neglected

Case 1: R_f is neglected

We will limit our analysis to short periods, i.e. $t << R_f\, C_f$.

In this range, the resistor R_f does not significantly affect the circuit and can be neglected (Fig. 13). On the other hand, we **do not assume that the amplifier gain A is infinite**, and will perform an exact analysis.

Note: A **negative gain** (-A) is important for **stability**—the feedback must be negative at low frequencies.

We use the **feedback analysis method** introduced in Lecture 5, applying **Mason's formula** and cutting the feedback path at the amplifier input:

$$A_{FB} = \frac{FF + A_{in} A_{ol}}{1 - \beta A_{ol}} \qquad (1)$$

For simplicity, we will **neglect the feedforward factor FF**.

On the right side of Fig. 14, the **test circuits** for calculating the factors A_{in}, A_{ol} and β are shown.

The **red circle** marks the test voltage (set to 1 V).

The **arrow** indicates the point where the voltage is measured; this measured voltage equals the value of the corresponding factor.

Typically, one voltage must be set to zero by **shorting to ground**, indicated by a **red line** in the figure.

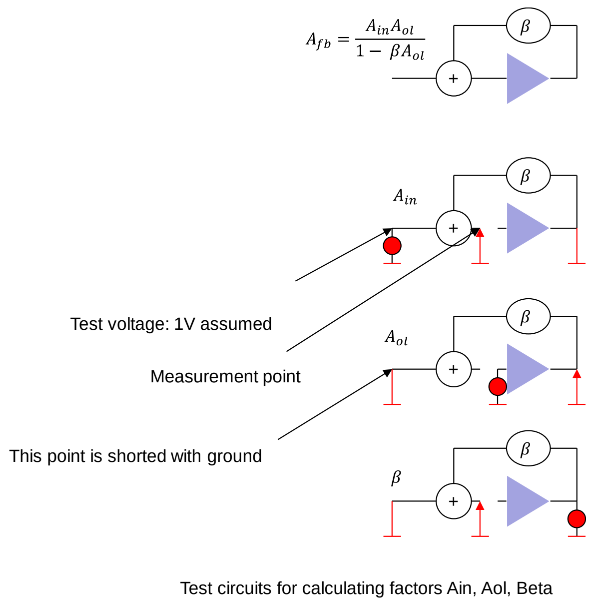


Fig 14: Test circuits for calculation of coefficients in Mason's formula

We will not neglect the input capacitance of amplifier, denoted with C_g in Fig 15.

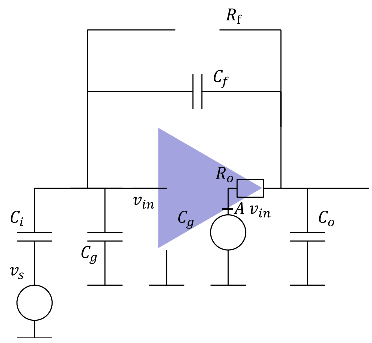


Fig 15: We take C_g into account

In the first step break the feedback loop v_{in} and v_{in}* as shown in Fig 16.

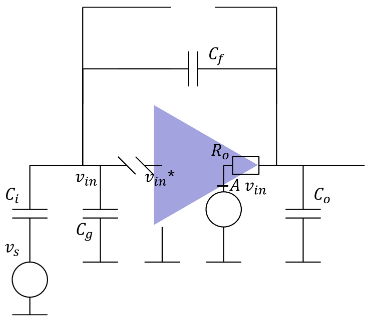


Fig 16: Step 1: feedback path is cut

Let us now calculate the coefficients A_{in}, A_{ol} and β.

Calculation of A_{in}

A_{in} can be derived using formula for voltage dividers, as illustrated in Fig 17. The result is:

$$A_{in} = \frac{sC_i}{s(C_i+C_g+C_f)} = \frac{sC_i}{s(C_i^+ + C_f)} \qquad (16)$$

with

$$C_i^+ = C_i + C_g$$

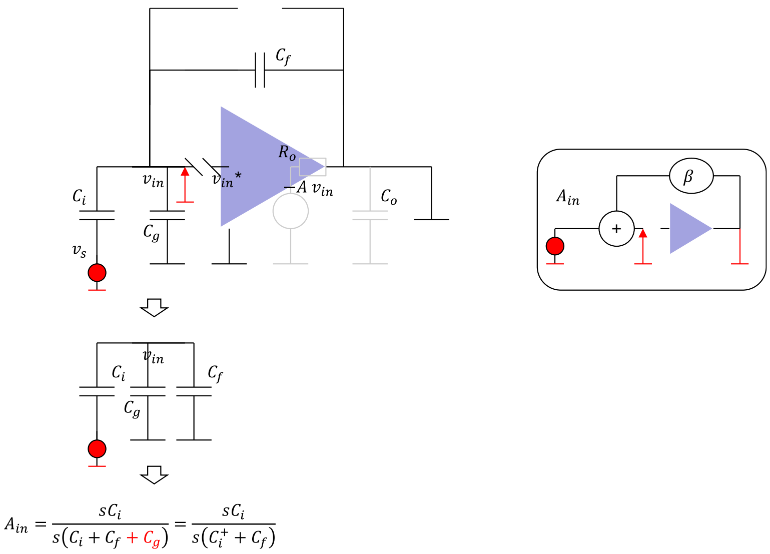


Fig 17: Calculation of A_{in}.

Calculation of A_{ol}

The formula for A_{ol} is a bit longer, Fig 18. The capacitors C_o, C_f and C_i^+ form an equivalent capacitance C_o', which is attached to R_o. R_o and C_o' form a low-pass filter.

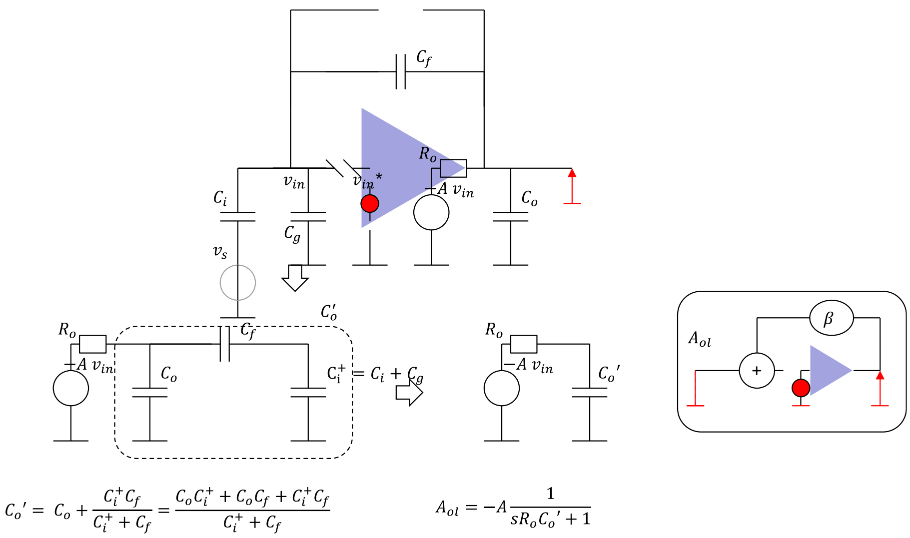


Fig 18: Calculation of A_{ol}.

Therefore:

$$A_{ol} = -A\frac{1}{sR_oC_o'+1} \quad (17)$$

How large is C_o'?

It is a parallel circuit of C_o and the series C_f and C_i^+.

It holds:

$$C_o' = C_o + \frac{C_i^+C_f}{C_i^++C_f} = \frac{C_i^+C_f+C_i^+C_o+C_oC_f}{C_i^++C_f} = \frac{\sum C_iC_k}{C_i^++C_f} \quad (18)$$

$\sum C_iC_k$ is the factor ($C_o C_i^+ + C_oC_f + C_i^+C_f$), the sum of all combinations of the capacitances.

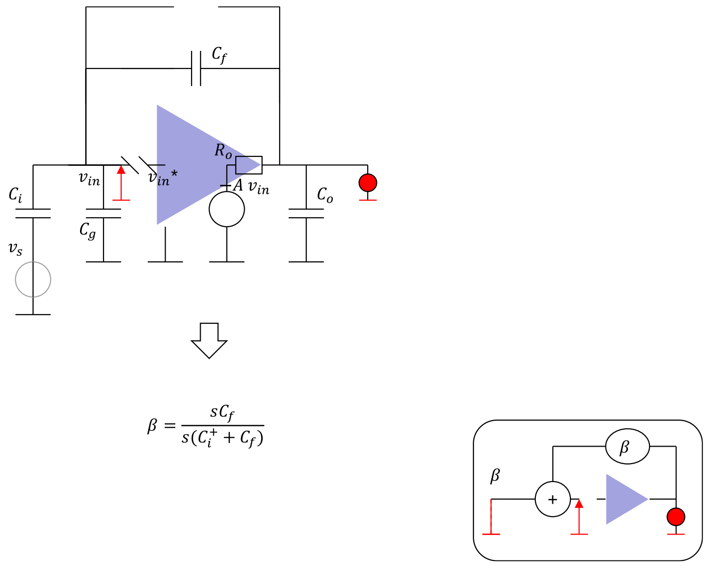


Fig 19: Calculation of β.

Calculation of β

Finally, we calculate β (Fig 19). Using formula for voltage divider, we can obtain:

$$\beta = \frac{sC_f}{s(C_i^++C_f)} \quad (19)$$

Final result

After calculation of all factors, we can apply the Mason's formula (1). We obtain:

$$A_{fb} = \frac{A_{in}A_{ol}}{1-\beta A_{ol}} = \frac{A_{in}}{\beta}\frac{\beta A_{ol}}{1-\beta A_{ol}} = -\frac{C_i}{C_f}\frac{\frac{\beta \cdot A}{(sT_0+1)}}{(1+\frac{\beta \cdot A}{(sT_0+1)})} = -\frac{C_i}{C_f}\frac{\beta \cdot A}{(sT_0+1+\ \beta \cdot A)}$$

$$= -\frac{C_i}{C_f}\frac{\beta \cdot A}{(1+\beta \cdot A)}\frac{1}{\left(\frac{sT_0}{1+\beta \cdot A}+1\right)}$$

$$= -\frac{C_i}{C_f}\frac{\beta \cdot A}{(1+\beta \cdot A)}\frac{1}{\left(\frac{sT_0}{\beta \cdot A}\ \frac{\beta \cdot A}{1+\beta \cdot A}+1\right)}$$

or:

$$A_{fb} = -\frac{C_i}{C_f}\alpha\frac{1}{\left(\frac{sT_0\alpha}{\beta\times A}+1\right)} \quad (20)$$

with

$$\alpha \equiv \frac{\beta\times A}{1+\beta\times A}$$

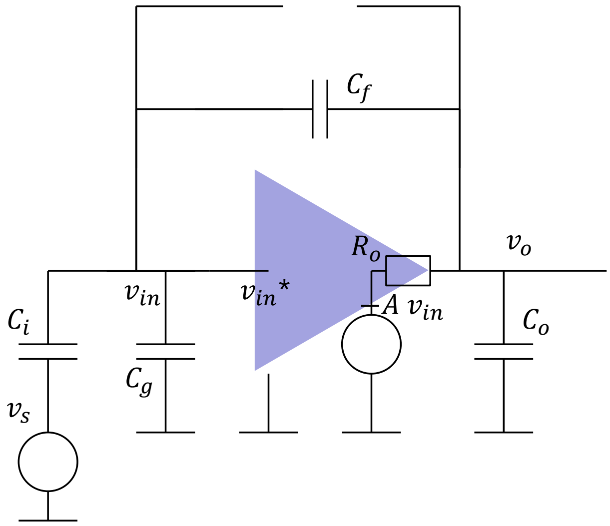
This is shown in Fig 20.

$$\beta = \frac{C_f}{C_i^+ + C_f}$$

$$A_{in} = \frac{C_i}{(C_i^+ + C_f)}$$

$$A_{ol} = -A\frac{1}{sT_o + 1}$$

$$T_o \equiv R_o C_o'; \; C_o' = C_o + \frac{C_i^+ C_f}{C_i^+ + C_f}$$



$$\Downarrow$$

$$A_{fb} = \frac{A_{in}A_{ol}}{1-\beta A_{ol}} = \frac{A_{in}}{\beta}\frac{\beta A_{ol}}{1-\beta A_{ol}} = \frac{C_i}{C_f}\frac{\frac{\beta\cdot A}{(sT_0+1)}}{\left(1+\frac{\beta\cdot A}{(sT_0+1)}\right)} = -\frac{C_i}{C_f}\frac{\beta\cdot A}{(sT_0+1+\beta\cdot A)}$$

$$= -\frac{C_i}{C_f}\frac{\beta\cdot A}{(1+\beta\cdot A)}\frac{1}{\left(\frac{sT_0}{1+\beta\cdot A}+1\right)} = -\frac{C_i}{C_f}\frac{\beta\cdot A}{(1+\beta\cdot A)}\frac{1}{\left(\frac{sT_0}{\beta\cdot A}\frac{\beta\cdot A}{1+\beta\cdot A}+1\right)}$$

Fig 20: Final calculation with Mason's formula

One can show the following

Without feedback ($C_f = 0$, $\beta = 0$) the amplifier gain is:

$$\mathbf{A_{fb}} = -\frac{C_i}{C_i+C_g}\cdot A\frac{1}{(sT_0+1)}$$

with the time constant

$$T_0 = R_o C_o'$$

With feedback, both the gain and time constant are reduced by the factor $1 + \beta\times A$:

$$\mathbf{A_{fb}} = -\frac{\frac{C_i}{C_i+C_g}\cdot A}{(1+\beta\cdot A)}\frac{1}{\left(\frac{sT_0}{1+\beta\cdot A}+1\right)}$$

Applying feedback reduces the gain but increases the bandwidth, such that the gain–bandwidth product is preserved.

Summary:

The formula (20) can be written in compact form:

$A_{fb} = -\frac{C_i}{C_f}\alpha\frac{v_s}{(sT_r+1)}$ (21)

Here we define the time constant with feedback:

$T_r = \frac{sT_0\alpha}{\beta\times A}$ (22)

T_0 is a time constant without feedback:

$T_0 = R_o C'_o$

with

$C'_o = C_o + \frac{C_i^+ C_f}{C_i^+ + C_f}$, $\beta = \frac{C_f}{C_i^+ + C_f}$, $\alpha \equiv \frac{\beta\times A}{1+\beta\times A}$, $C_i^+ = C_i + C_g$

The formula (21) is the transfer function of the low-pass filter with the gain C_i/C_f.

The transfer function of our amplifier has a low-pass behavior.

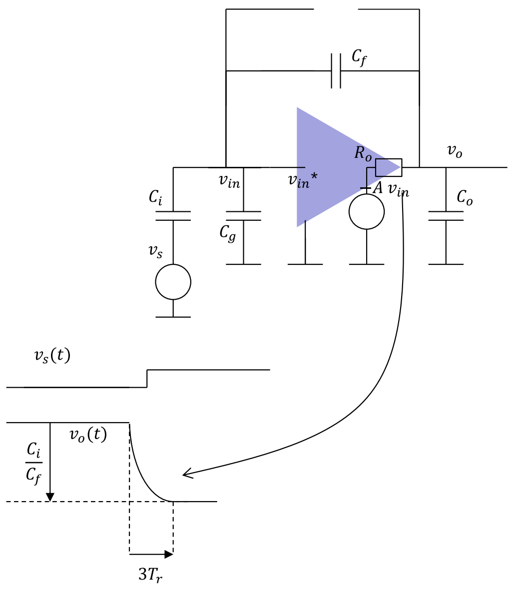


Fig 21: Time constant in the step response

The step response is shown in Fig 21.

The time constant T_r determines the rise time of the output signal. After about $3 \times T_r$ the output signal reaches 95% of its final amplitude - C_i/C_f. If we want our amplifier to be faster, we should choose an amplifier with a larger open-loop gain A, which reduces the effective time constant.

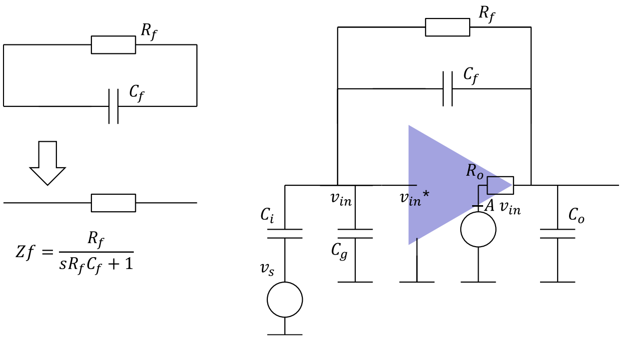


Fig 22: Influence of R_f to step response

Case 2: R_f is not neglected

How does the behaviour of the circuit change if we take R_f into account?

The formula without R_f is (22):

$$v_o = \frac{C_i}{C_f}\frac{1}{1+sT_r}$$

It can be rewritten as:

$$v_o = \frac{Z_f}{Z_{in}}\frac{1}{1+sT_r} \quad (21)$$

We can now substitute the equation for impedance of the parallel connection of R_f and C_f

$Z_f = R_f/(sC_fR_f + 1)$

into equation (24), giving:

$$v_o = -\frac{Z_f}{Z_i}\frac{v_s}{1+sT_r} = -\frac{sR_fC_i}{1+sT_f}\frac{v_s}{1+sT_r} \qquad (22)$$

With R_f the amplifier is a combination of high-pass and low-pass filter. For proper operation, it must hold that $T_f = R_fC_f >> T_r$, otherwise the gain of the circuit would be reduced.

If we draw the Bode plot of this transfer function, the band-pass behaviour of the amplifier becomes clearly visible, as shown in Fig 23.

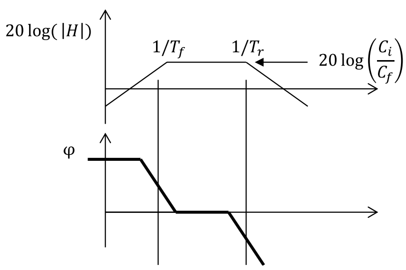


Fig 23: Inverting amplifier with R_f und C_f – frequency response

The step response is shown in Fig 24.

As first, the output voltage increases almost to the level $-C_i/C_f$, then it drops again at zero. The rise time is approximately $3 \times T_r$ and the fall time $3 \times T_f$.

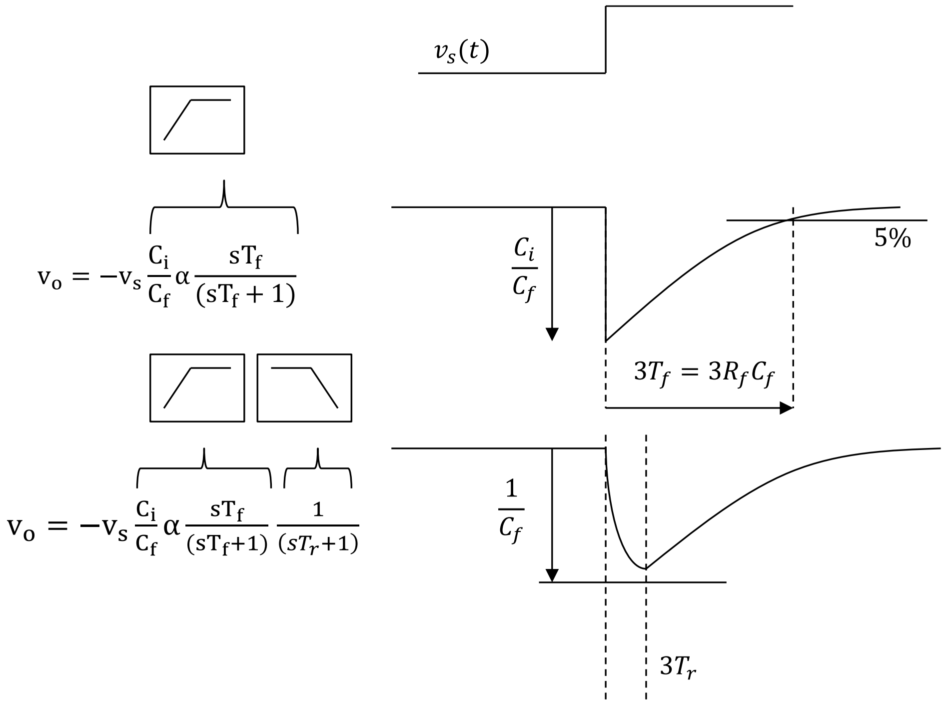


Fig 24: Inverting amplifier with R_f und C_f – step response

Transistor implementation of the amplifier

At the end of this lecture, we discuss how the amplifier can be implemented using a transistor. A common realization is with a MOSFET and an output resistance. The small-signal model of the amplifier is shown in Fig 25. It can be represented as a voltage-controlled current source, where the amplification is given by the transconductance (g_m). As mentioned earlier, it is important that a sufficiently large DC bias current flows through the transistor. Only under this condition does the transistor achieve a high transconductance, ensuring effective amplification.

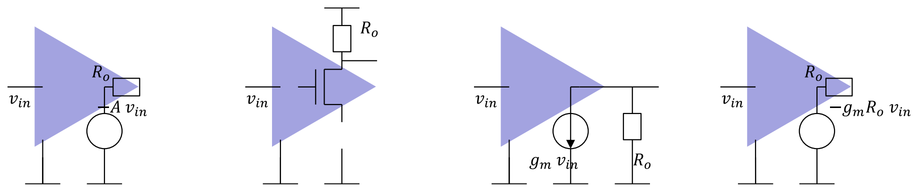


Fig 25: Voltage amplifier implemented with a transistor

We can convert the current source that models the transistor into an equivalent voltage source, as shown in Fig 25. This is useful because it allows us to apply the same formulas we derived earlier for voltage amplifiers, simplifying the analysis.

It holds

$$A = g_m R_o$$

And

$$R_o = R_o$$

If we replace this in the formulas for A_{fb} (21) and T_r (22) we get

$$A_{FB} = -\frac{C_i}{C_o}\alpha\frac{1}{1+sT_r} \quad (23)$$

and

$$T_r = \frac{sC'_o\alpha}{\beta g_m} \quad (24)$$

Interestingly, the time constant T_r does not depend on the R_{out}.

Influence of design parameters to gain and time constant

Let us summarize how the design parameters C_i, C_f, C_o and g_m influence the amplifier in terms of gain and time constant.

When C_i increases, the amplification with feedback increases but the amplifier becomes slower. This follows from equations (26) and (27) and is illustrated in Fig 26.

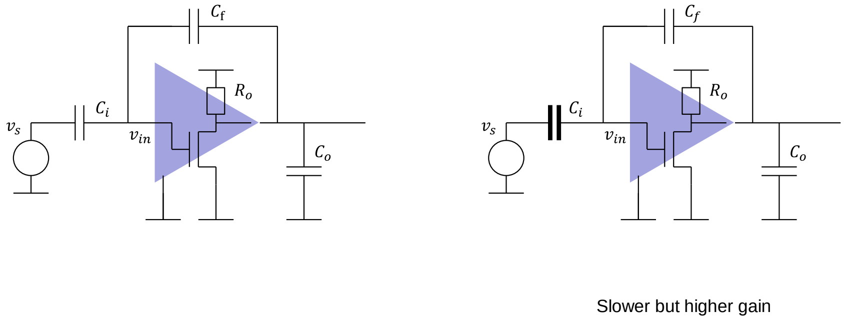


Fig 26: How does C_i influence step response?

If C_f increases, the amplification with feedback becomes smaller. The amplifier becomes faster until the time constant reaches its minimum value C_o/g_m. This is shown in Fig 27.

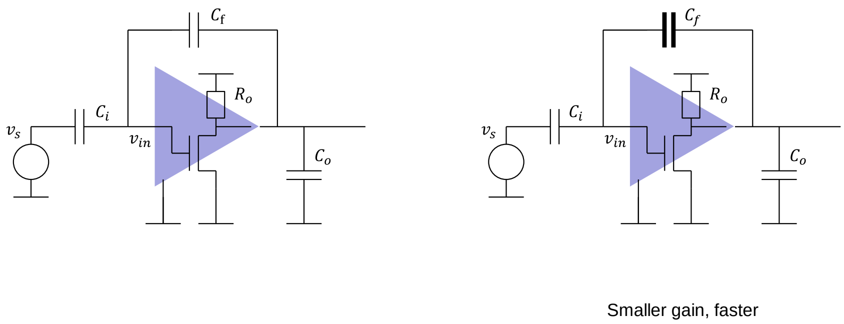


Fig 27: How does C_f influence step response?

A larger load capacitance C_o makes the amplifier slower, as shown in Fig 28.

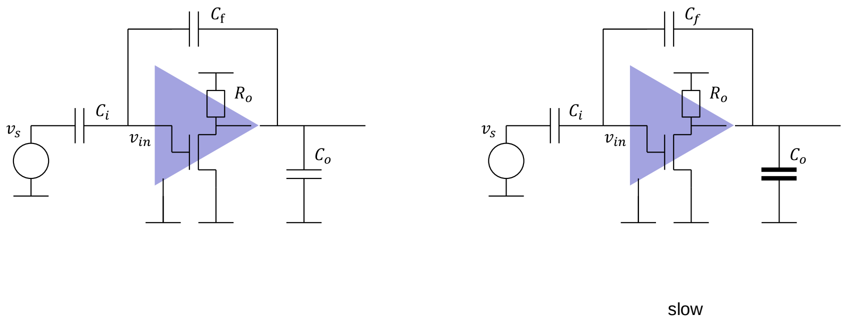


Fig 28: How does C_o influence step response?

How can we address this?

We can connect **several amplifiers in parallel** (Fig 29). In this configuration, the equivalent transconductance g_m increases, which reduces the time constant T_r, as can be seen from equations (26) and (27).

Important note: Simply connecting multiple transistors in parallel is **not sufficient**, because each transistor would then have a smaller bias current, and the effective g_m would not increase proportionally. To achieve the desired improvement, it is necessary to **increase both the transistor size and the bias current**, which is accomplished by the parallel connection.

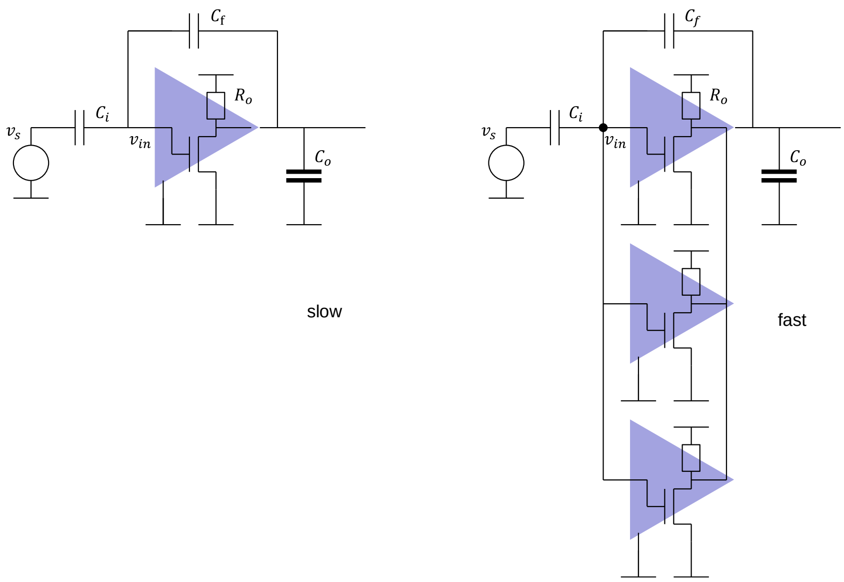


Fig 29: Use of parallel connected amplifiers make the step response faster

Charge-sensitive amplifier

Charge amplifiers are frequently used to amplify **sensor signals**. This circuit has a structure **very similar** to the voltage amplifier discussed in this lecture.

In this section, we will derive the **transfer function** of the charge amplifier.

We begin by reviewing the **voltage amplifier** as a starting point for understanding the charge amplifier.

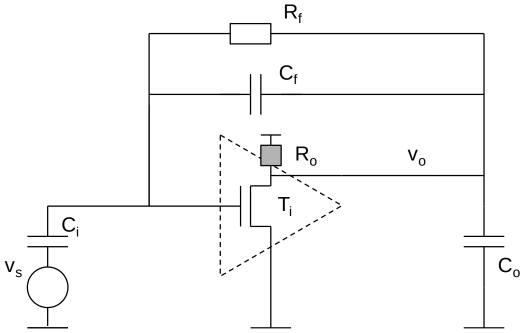


Figure 30: Voltage amplifier

The transfer function of the voltage amplifier is (24):

$$v_o = -v_s \frac{C_i}{C_f} \alpha \frac{sT_f}{(sT_f+1)} \frac{1}{(sT_r+1)} \qquad (1B)$$

with

$$T_f = R_f C_f, \; T_r = \frac{sT_0\alpha}{\beta \times A}, \; T_0 = R_o C'_o, \; \beta = \frac{C_f}{C_i^+ + C_f}, \alpha \equiv \frac{\beta \times A}{1+\beta \times A}, C_i^+ = C_i + C_g$$

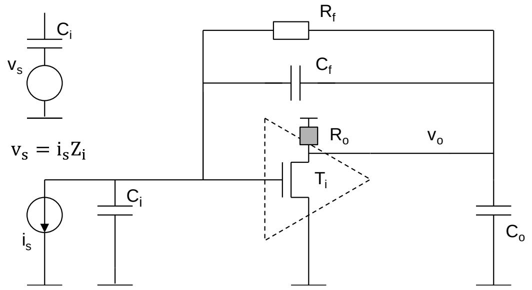


Figure 31: Converting the voltage source to a current source

We first convert the voltage source at the input into a current source:

$$v_s = i_s Z_i \qquad (2B)$$

Here, i_s the current of the current source, and the input capacitance C_i is usually the **sensor capacitance**.

Current Gain Derivation

The current gain is defined as v_o / i_s.

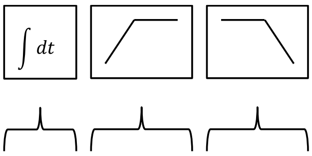
Starting from (1B):

$$v_o = -v_s \frac{C_i}{C_f} \alpha \frac{sT_f}{(sT_f+1)} \frac{1}{(sT_r+1)}$$

Using (2B), we have:

$$v_o = -\frac{i_s}{sC_i} \frac{C_i}{C_f} \alpha \frac{sT_f}{(sT_f+1)} \frac{1}{(sT_r+1)} = -\frac{i_s}{sC_f} \alpha \frac{sT_f}{(sT_f+1)} \frac{1}{(sT_r+1)} \qquad (3B)$$

Important note: The gain is independent of C_i if βA is >> 1. This is useful because the sensor capacitance may be large and unknown. The resulting gain v_o/i_s is a combination of an integrator, a high-pass filter, and a low-pass filter, as illustrated in Figure 32.



$$v_o = -\frac{i_s}{sC_f} \alpha \frac{sT_f}{(sT_f+1)} \frac{1}{(sT_r+1)}$$

Figure 32: Transfer function of the charge amplifier

The **impulse response** to a current pulse with **total charge** Q is shown in Figure 33.

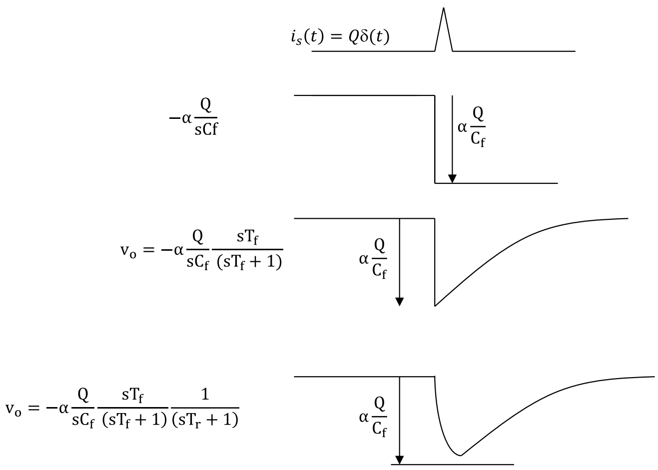


Figure 33: Impulse response of the charge amplifier to a current pulse with the integral Q

Amplitude of the output signal depends on charge Q divided by C_f! The charge is linearly amplified, meaning the output signal is proportional to Q and independent of the form of the input pulse. For this reason, we call the circuit charge-sensitive amplifier.

Method for Fast Determination of Coefficients in the Characteristic Polynomial (optional)

A circuit with **n independent capacitors (**Figure 34**)** has a **polynomial of degree n** in the denominator of its transfer function.

$$u_{out}(t)(or\ i_{out}(t)) = \frac{b_n D^n + \cdots + b_1 D + 1}{a_n D^n + \cdots + a_1 D + 1} u_G(t) \quad (1C)$$

D = d/dt is the differential operator, u_G is the input generator voltage.

Each independent capacitor can have **any voltage** and **any initial condition**.

The polynomial in the numerator has a degree **less than or equal to n**.

The **denominator polynomial**

$$a_n D^n + \cdots + a_1 D + 1 \quad (2C)$$

determines the **natural behavior** of the circuit, i.e., how the circuit discharges when the sources are turned off.
The **numerator polynomial** (together with the denominator) determines the **switch-on behaviour** of the circuit

Calculation of Coefficients

Coefficient a_1 of the denominator polynomial 2C is calculated as:

$$a_1 = \sum C_i R_i^0$$

Where R^0_i is the resistance "seen" by C_i (Figure 34).

Coefficient a_2 of 2C is calculated as:

$$a_2 = \sum C_i C_j R_i^0 R_j^i$$

where R^i_j is the resistance seen by C_j when C_i is short-circuited (Figure 35).

In practice, only these **two coefficients** are important, because they determine the **dominant** and the **second time constant**:

The **dominant time constant** $\tau_1 = a_1$ determines how fast the **step response** is and how large the **system bandwidth** is.

The **ratio** a_2/a_1 **is the second time constant** and it determines whether a circuit is **stable** when feedback is used. The second time constant a_2/a_1 must be **much smaller than a_1**.

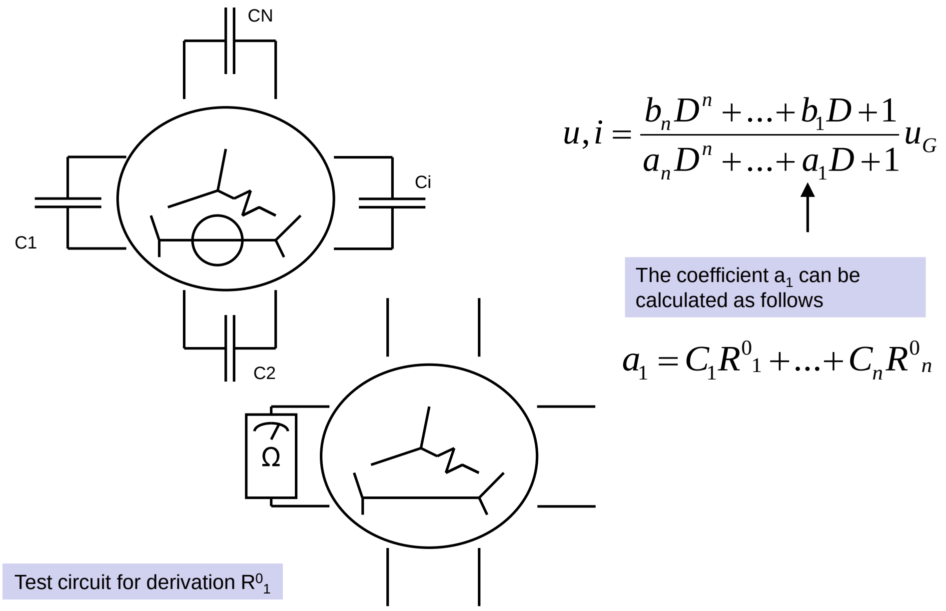


Figure 34: Fast calculation of coefficients in the characteristic polynomial.

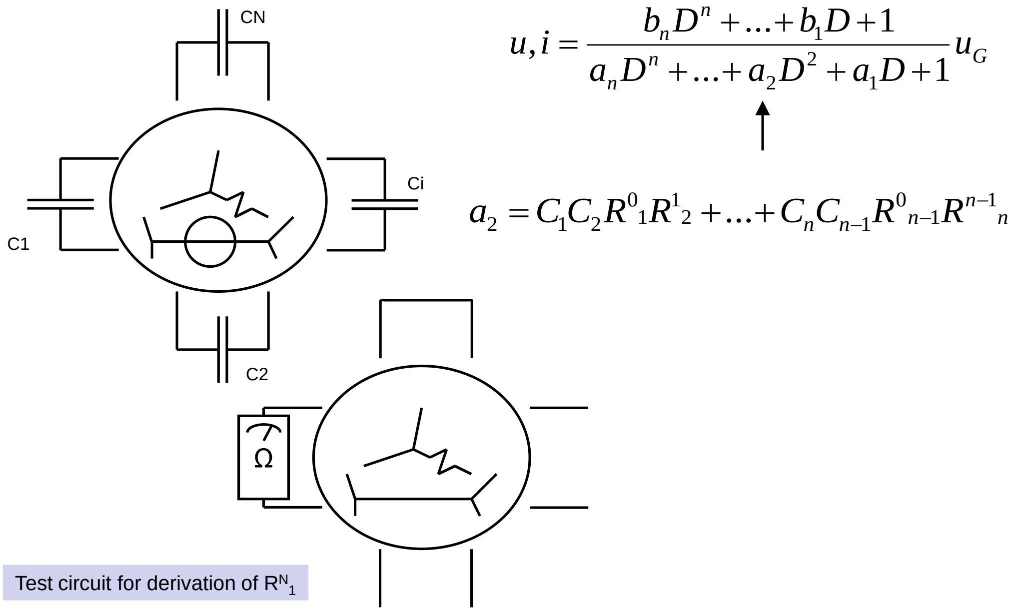


Figure 35: Fast calculation of coefficients in the characteristic polynomial.