Lecture 10 and 11

The topics of this lecture are:

- Noise Theory and Thermal Noise in Resistor
- Noise in the MOS transistor
- Noise in the voltage amplifier and the charge sensitive amplifier
- 1/f noise and transistor mismatch

Electronic noise refers to the observable fluctuations in current or voltage in an electrical system.
To calculate the amplitude of a noise signal, it is necessary to identify and mathematically describe both the **noise source** (the input variable) and the **transfer function** of the measurement device (the circuit connected to the noise source).

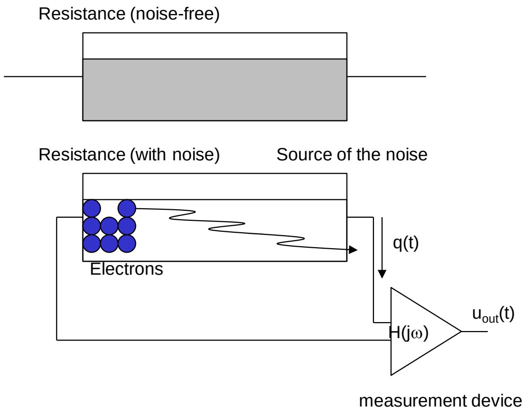
If the circuit is time-invariant, its transfer function can be represented as a frequency-dependent complex function H(s) defined as the ratio of the Fourier- or Laplace-transformed output and input signals:

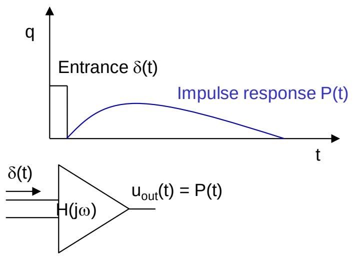
$H(s) = V_{out}(s)/V_{in}(s)$

Alternatively, the circuit can be characterized by its **impulse (pulse) response P(t)**. This representation is particularly useful when the circuit is time-dependent, for example when it is switched on at a specific moment.

Thermal Noise in a Resistor

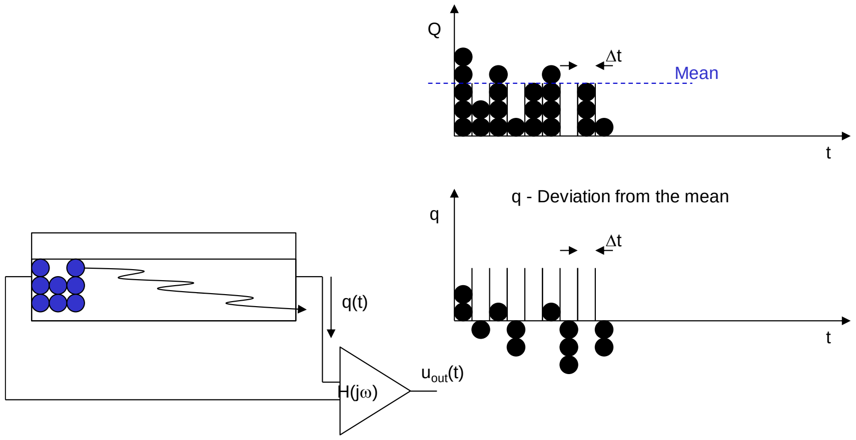
Electrons move through a resistive material not only due to the applied electric field (drift current), but also because they possess kinetic (thermal) energy. This random thermal motion causes small fluctuations in the electric current, which are observed as **noise**.





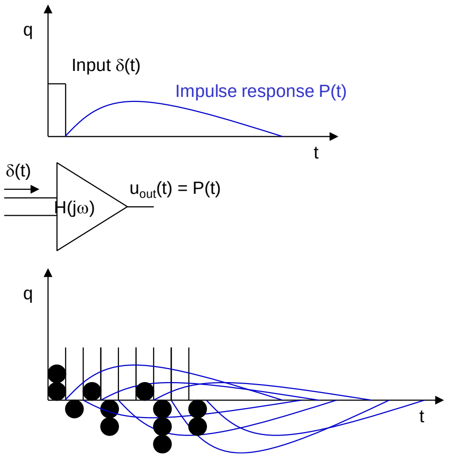
We divide (quantize) time into short intervals Δt. During each interval, a certain number of electrons leave the resistor. The **mean transferred charge** ⟨Q⟩ is equal to the average current multiplied by Δt.

Due to noise, the actual transferred charge fluctuates around this mean value. These deviations from the mean are denoted by q.



The noise charges q occurring in short time intervals Δt can be interpreted as **charge pulses**. We assume that the resistor is connected to a measurement device, which is characterized by a **pulse (impulse) response** and an associated **transfer function**.

Our goal is to calculate the resulting noise signal at the output, u_{out}. The output signal u_{out} is generated by the superposition of the noise charge pulses q produced by the resistor and shaped by the transfer characteristics of the measurement device.



Each input pulse generates a corresponding output signal.

The output signal produced by a noise charge q(τ) at time τ is given by the product of the charge and the pulse (impulse) response:

$$u_{out,\tau}(t) = q(\tau)P(t,\tau)$$

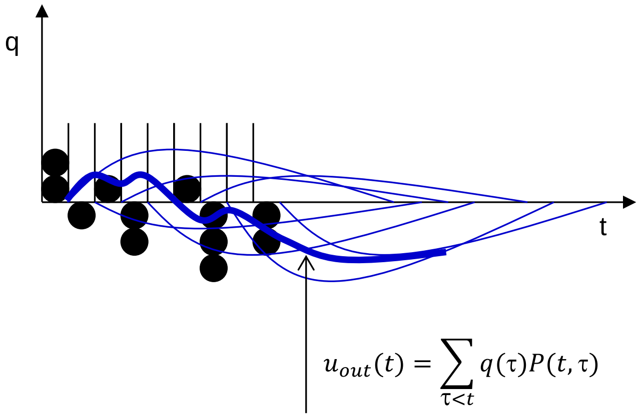
Where P(t, τ) is the pulse response at a moment t to an input pulse with integral 1 (Dirac pulse) that was generated at the moment τ.

The total amplitude of the output noise signal at time t is given by the sum of all output signals generated in the past:

$$u_{out}(t) = \sum_{\tau<t} q(\tau)P(t,\tau)$$

Since the actual values of the noise charges q(τ) are unknown and random, only their statistical properties—such as the mean and variance—can be determined.

Consequently, it is not possible to calculate the instantaneous value of the output noise signal. Instead, we calculate the **variance** $u_{out}(t)$, which corresponds to the **noise power** at the output.



Starting from

$$u_{out}(t) = \sum_{\tau<t} q(\tau)P(t,\tau)$$

the following expression for the noise in the time domain can be derived in the limit Δt→0

$$\langle u(t)^2\rangle = \sum_{\tau<t} \langle q^2\rangle (P(t,\tau))^2 = \int_{-\infty}^{t} \frac{\langle q^2\rangle}{\Delta t}(P(t,\tau))^2\, d\tau$$

Here, Δt is the length of the time intervals, and

P(t,τ) denotes the pulse (impulse) response at time ttt due to an input pulse generated at time τ.

This time-domain noise formulation is particularly suitable for **time-varying systems**, such as circuits containing switches or other elements whose properties change with time.

If the system is **time-invariant**, the noise can also be analyzed in the **frequency domain**. This derivation proceeds in two steps.

Step 1: Time-invariant pulse response

For a time-invariant system, the pulse response depends only on the time difference:

$P(t,\tau) = P(t-\tau)$ and $P(u) = 0$, for $u < 0$

Starting from the time-domain expression,

$$\langle u(t)^2\rangle = \int_{-\infty}^{t} \frac{\langle q^2\rangle}{\Delta t}(P(t,\tau))^2\, d\tau = \int_{-\infty}^{t} \frac{\langle q^2\rangle}{\Delta t}(P(t-\tau))^2\, d\tau$$

we substitute u=t−τ and obtain

$$\int_{0}^{\infty} \frac{\langle q^2\rangle}{\Delta t}(P(u))^2\, du$$

Step 2: Transformation to the frequency domain (Parseval's theorem)

Using Parseval's theorem

$$\int_{-\infty}^{\infty} (P(t))^2\, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |P(i\omega)|^2\, d\omega$$

we obtain

$$\langle u^2 \rangle = \int_{0}^{\infty} \frac{\langle q^2 \rangle}{\Delta t} (P(u))^2\, du = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\langle q^2 \rangle}{\Delta t} |P(i\omega)|^2\, d\omega = \int_{0}^{\infty} \frac{2\langle q^2 \rangle}{\Delta t} |P(i\omega)|^2\, df$$

Introducing the **noise power spectral density** S(f), this can be written as

$$\langle u^2 \rangle = \int_{0}^{\infty} S(f) |H(i\omega)|^2\, df$$

where H(iω) is the transfer function of the system.

White noise

The function S(f) represents the **spectral power density of the noise source**. For **white noise**, it holds that

$$S(f) = 2\frac{\langle q^2 \rangle}{\Delta t}$$

When S(f) is independent of both Δt and frequency, the noise is called *white*. In this case, the noise at the input can be modeled as arbitrarily short pulses (Dirac delta pulses). The Fourier transform of these pulses is constant, meaning that the noise contains all frequencies—analogous to white light, which contains all wavelengths.

Colored noise

If the noise pulses at the input have a finite temporal shape, the spectral density becomes frequency-dependent, and the noise is no longer white. In this case,

$$\mathrm{S(f)} = 2\frac{\langle \mathrm{q}^2 \rangle}{\Delta \mathrm{t}} |\mathrm{P_{in}(i\omega)}|^2$$

A common example of colored noise is **1/f noise**, whose spectral density decreases with increasing frequency.

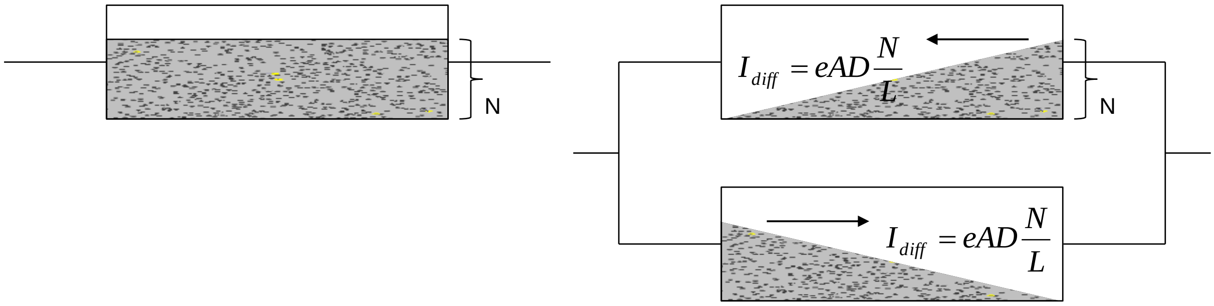
How large is the spectral power density of the thermal noise generated by a resistor? To answer this question, we must determine the variance of the noise charge fluctuations q. In the following, we show how this variance can be calculated.

For alternative derivation see:

Johnson–Nyquist noise - Wikipedia

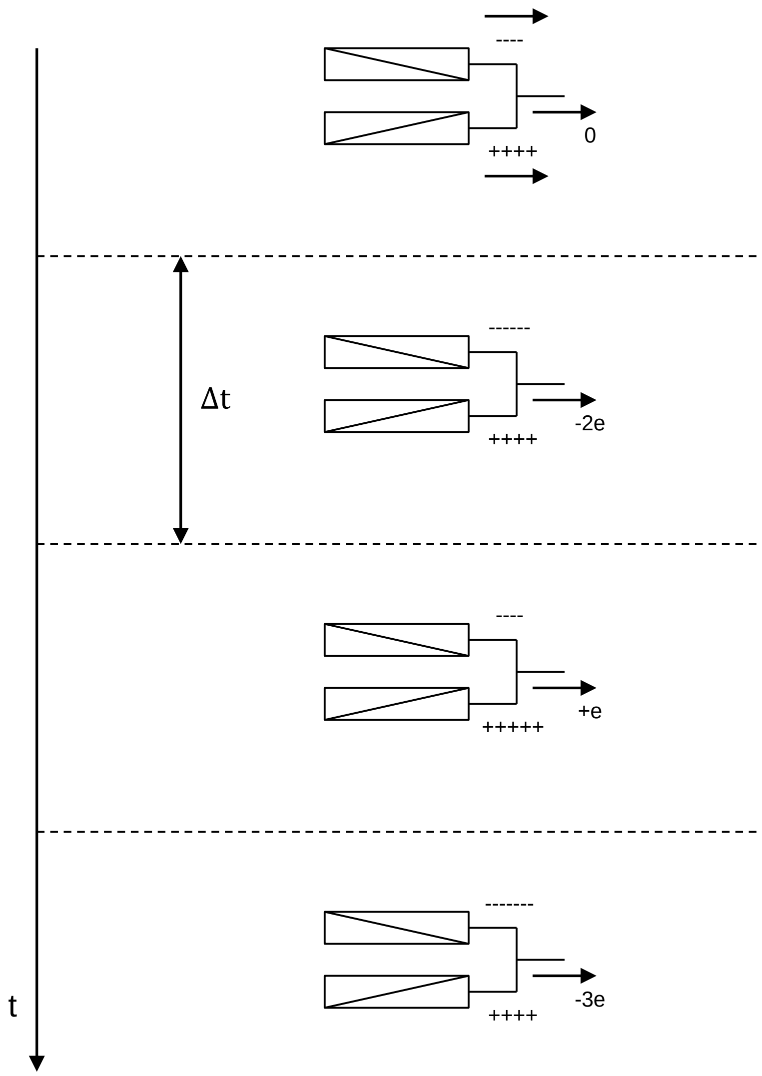
We conceptually divide the resistor into two parallel partial resistors, each with a uniformly varying charge density. In each partial resistor, a **diffusion current** flows due to the charge density gradient. The total resistance of the parallel connection is R.

The diffusion currents in the two partial resistors have equal magnitude but flow in opposite directions. As a result, the **total diffusion current is zero**.



The **mean number of electrons** leaving each partial resistor in a time interval Δt is

$\langle n\rangle = I_{diff}\Delta t / e$, where I_{diff} is the diffusion current and e is the electron charge. The **actual number of electrons** fluctuates randomly around this mean due to thermal motion.



To determine the variance of the noise charges, we need the **distribution function** of the number of electrons. Because electrons are discrete particles, the fluctuations in the number of electrons n leaving a resistor part in a time interval Δt can be described by a **Poisson distribution**:

$$P_{\lambda}(n) = \frac{\lambda^{n}}{n!} e^{-\lambda}; n = 0,1,2 \ldots; \lambda = \langle n \rangle$$

Where $\lambda=\langle n \rangle$ is the mean number of electrons.

In our case:

$P_{\langle n \rangle}(n) = \frac{\langle n \rangle^{n}}{n!} e^{-\langle n \rangle}$; n is the number of electrons

A key property of the Poisson distribution is that its **variance equals its mean**

Since the charge q associated with n electrons is q=en, the variance of q is:

$$< q^2 > \ = e^2 < n^2 > \ = \ e^2 < n > \ = eI_{\mathrm{diff}} \Delta t$$

The total variance doubles because there are **two parallel partial resistors**, giving:

$$< q_{total}{}^2 > \ = 2eI_{\mathrm{diff}} \Delta t$$

Spectral power density of thermal noise

The noise spectral density is related to the variance by:

$$\mathrm{S(f)} = 2\frac{\langle \mathrm{q}^2 \rangle}{\Delta \mathrm{t}} = 4\mathrm{eI}_{\mathrm{diff}}$$

Using the expressions for diffusion current in terms of electron density N, cross-section A, resistor length L, and diffusion constant D:

$$\mathrm{I}_{\mathrm{diff}} = \frac{\mathrm{DeNA}}{\mathrm{L}} 0 \Rightarrow \mathrm{S(f)} = \frac{4\mathrm{e}^2\mathrm{DNA}}{\mathrm{L}}$$

Applying the **Einstein relation** D/μ=kT/e and the relation between resistance and mobility:

$$R = \frac{L}{e\mu NA}$$

we obtain the standard **Johnson–Nyquist formula** for thermal noise:

$$\mathrm{S}_{\mathrm{IR}}(\mathrm{f}) = 4\mathrm{e}^2\mathrm{DN}\frac{\mathrm{A}}{\mathrm{L}} = 4\mathrm{kTe\mu N}\frac{\mathrm{A}}{\mathrm{L}} = 4\mathrm{kT/R}$$

Conclusion: Thermal noise arises from the combination of **random thermal motion of electrons** (diffusion currents) and the **discrete nature of electric charge** (Poisson statistics).

$$\mathrm{S}_{\mathrm{IR}}(\mathrm{f}) = 4\mathrm{kT/R}$$

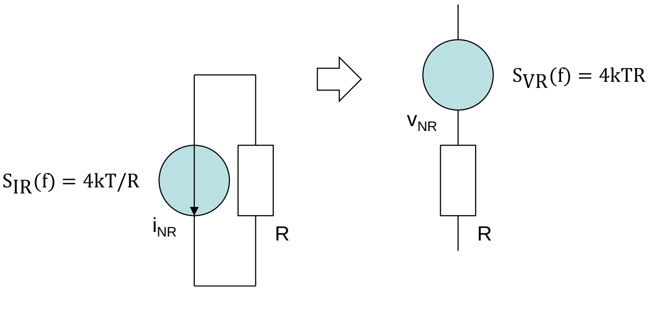
The noise of a resistor can be described with a current source with spectral power density S_{IR} = 4kT/R. The unit of S_{IR} is A^2s.

We can convert the current source with a parallel resistor into a voltage source with a series resistor. For the voltage and current of the respective sources, hold:

U = I x R

The power density of the voltage source is

$S_{VR} = S_{IR} \times R^2$



Noise in MOS transistor

The MOSFET channel can be modeled as a **resistor with a non-uniform charge carrier density** along its length.

The **thermal noise** of the channel can be represented by a **current source** in parallel with the channel. Its **power spectral density** can be calculated using the resistor formula:

$$S_{IT} = 4kT\frac{1}{<R>} \qquad (1)$$

1 / <R> is the average conductance of the channel:

For a MOSFET channel of length L and width W, with carrier mobility μ, the average conductance is:

$$1/\langle R\rangle = \frac{\mu W<Q'>}{L} \qquad (2)$$

<Q‘> is the **average charge per unit area** (unit C/m^2) along the channel. This average can be calculated as:

$$< Q' >= \frac{1}{L}\int_0^L Q'(x)dx \qquad (3)$$

with Q′(x) being the local charge density at position x along the channel

The formula (2) applies because:

$$\frac{1}{\langle R\rangle} = \mu e\frac{A}{L} < n > \quad = \mu\frac{W}{L} < ten > \quad = \quad \frac{\mu W < Q' >}{L}$$

n is the density of electrons, μ mobility, e charge of the electron, A cross-section, t depth and L length of the channel.

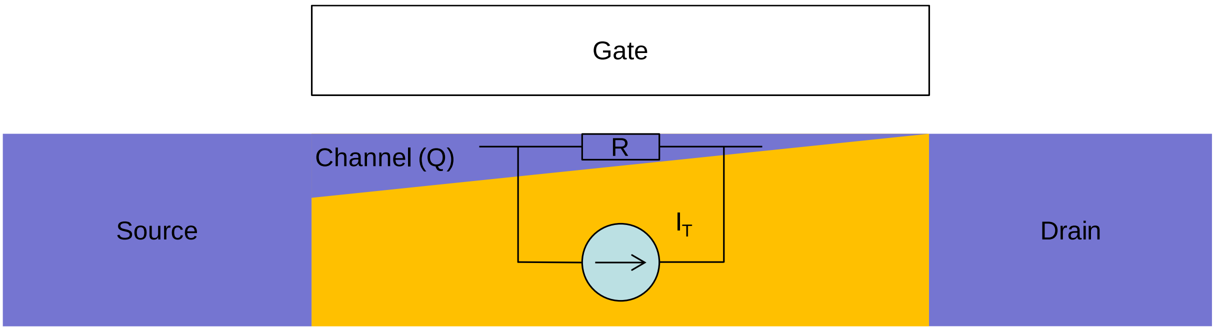


Figure 1: Thermal noise in the MOSFET channel

Substituting (3) into (2) and (1), we obtain the formula for the power spectral density:

$$S_{IT} = 4kT\mu\frac{W}{L}\langle Q'\rangle = \frac{4kTW\mu}{L^2}\int_0^L Q'(x)dx \qquad (\ 1\ )$$

Average charge density along the channel

For a MOSFET in saturation, the average charge per unit area can be calculated as:

$$< Q' >= \frac{2}{3}C'_{ox}(V_{gs} - V_{th}) \qquad (5)$$

where C_{ox}' is the oxide capacitance per unit area, V_{GS} is the gate-source voltage, and V_{TH} is the threshold voltage.

(The exact derivation of equation (5) is in the next paragraph.)

Final formula for MOSFET channel thermal noise

Inserting the average charge (5) into the noise formula (4) gives:

$$S_{IT} = 4kT\mu\frac{W}{L}\frac{2}{3}C'_{ox}(V_{gs} - V_{th}) \qquad (6)$$

This result describes the **thermal noise current** of a MOSFET channel, accounting for the **linear variation of charge along the channel**.

Substituting (5) into (4), we get

$$S_{IT} = 4kT\mu\frac{W}{L}\frac{2}{3}C'_{ox}(V_{gs} - V_{th}) \qquad (2)$$

MOSFET Thermal Noise and Transconductance

The **transconductance** of a MOSFET in saturation is defined as:

$$g_m = \frac{dI_{dssat}}{dV_{gs}}$$

For a MOSFET in the saturation region (with parameter nnn for the slope factor):

$$I_{dssat} = \frac{1}{2n}\mu C'_{ox}\frac{W}{L}\left(V_{gs} - V_{th}\right)^2$$

Differentiating with respect to V_{GS} gives:

$$g_m = \frac{dI_{dssat}}{dV_{gs}} = \frac{d}{dV_{gs}}\left(\frac{1}{2n}\mu C'_{ox}\frac{W}{L}(V_{gs} - V_{th})^2\right) = \mu C'_{ox}\frac{W}{Ln}(V_{gs} - V_{th}) \qquad (7)$$

Substituting the formula for g_m (7) into that for S_{IT} (6), we obtain the final equation for the power spectral density of the current noise source in a MOSFET:

$$S_{IT} = 4kTn\frac{2}{3}g_m \qquad (3)$$

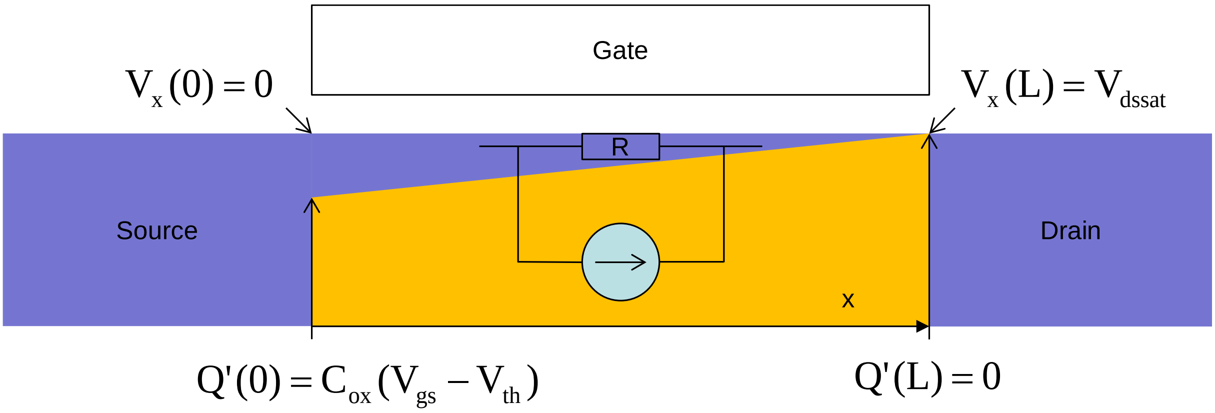


Figure 2: Thermal noise in the MOSFET channel

Derivation of the formula for transistor current I_{ds} and the average channel charge (optional)

In lecture 2 we had the following formula for the channel charge per area:

$$Q' = C'_{ox}\left(V_{gs} - V_{th}\right) \qquad (4)$$

The formula is valid in strong inversion for small V_{ds}.

The formula must be adjusted for larger V_{ds}. The channel charge per area varies from the value given by (9) on the source side to zero on the drain side, where the channel pinches off (Figure 2).

Let us denote the potential in the channel region (at the boundary between silicon and gate oxide) as $V_x(x)$. Coordinate x goes from source to drain (Figure 2). Let us define V_x with respect to the source potential V_s. That means:

$$V_x(0) = 0$$

and

$$V_x(L) = V_{ds}$$

The channel charge per area at any point x is:

$$Q'(x) = C_{ox}\left(V_{gs} - V_{th} - nV_x\right) \qquad (5)$$

Or simplified

$$Q'(x) = C_{ox}\left(V_{gst} - nV_x\right);\ V_{gst} \equiv V_{gs} - V_{th} \qquad (6)$$

This paragraph is a repeat from Lecture 3.

First, let us derive the equation for I_{ds} current for any V_{ds}.

We start with the formula for drift current:

$$I_{ds} = \mu\, W\, Q'(x)\, |E_x| = \mu\, C'_{ox}\, W \left(V_{gst} - nV_x\right) |E_x| \qquad (7)$$

E_x is the E-field component in the x-direction, W is the gate width, μ is the mobility of the charge carriers.

The following applies:

$$|E_x| = \frac{dV_x}{dx} \qquad (8)$$

Substituting (13) into (12), we get:

$$I_{ds} = \mu\, C'_{ox}\, W \left(V_{gst} - nV_x\right) \frac{dV_x}{dx}$$

or

$$I_{ds} dx = \mu\, C'_{ox}\, W \left(V_{gst} - nV_x\right) dv_x \qquad (9)$$

We can integrate the two sides:

$$\int_0^L I_{ds} dx = \int_0^{V_{ds}} \mu\, C'_{ox}\, W \left(V_{gst} - nV_x\right) dv_x$$

It follows:

$$I_{ds} = \frac{1}{2n} \mu\, C'_{ox} \frac{W}{L} \left[{V_{gst}}^2 - \left(V_{gst} - nV_{ds}\right)^2\right]$$

After squaring the second term, we get the general formula for transistor current:

$$I_{ds} = \mu\, C'_{ox}\frac{W}{L}\left[V_{gst}V_{ds} - n\frac{V_{ds}^{2}}{2}\right] \qquad (\ 10\ )$$

In saturation:

$$V_{ds} = V_{dssat} = \frac{V_{gst}}{n} \qquad (\ 11\ )$$

Substituting this into (15), we get the formula for saturation current:

$$I_{dssat} = \frac{1}{2n}\mu\, C'_{ox}\frac{W}{L}\, V_{gst}^{2} \qquad (\ 12\ )$$

Now let us calculate the average channel charge in a transistor in saturation:

$$\langle Q' \rangle = \frac{1}{L}\int_0^L Q'(x)dx$$

Let us start with the integral:

$$\int_0^L Q'(x)dx = \int_0^L C'_{ox}\left(V_{gst} - nV_x\right)dx$$

The integral is the total charge in the channel divided by W.

Since we do not know the function $V_x(x)$, we change the integration variable from x to Vx (substitution):

$$\frac{Q}{W} = \int_0^L C'_{ox}\left(V_{gst} - nV_x\right)dx = \int_0^L C'_{ox}\left(V_{gst} - nV_x\right)\frac{dV_x}{dV_x/dx} = \int_0^{V_{dssat}} C'_{ox}\left(V_{gst} - nV_x\right)\frac{dV_x}{|E_x|}$$

The E field can be calculated using (12):

$$I_{ds} = \mu\, C'_{ox}\, W\left(V_{gst} - nV_x\right)|E_x|; \Rightarrow |E_x| = \frac{I_{ds}}{\mu C'_{ox}\, W\,(V_{gst} - nV_x)} \qquad (\ 13\ )$$

It follows:

$$\frac{Q}{W} = \int_0^{V_{ds}} C'_{ox}\left(V_{gst} - nV_x\right)\frac{dV_x}{|E_x|} = \frac{\mu (C'_{ox})^2 W}{I_{ds}}\int_0^{V_{dssat}}\left(V_{gst} - nV_x\right)^2 dV_x$$

$$= \frac{\mu (C'_{ox})^2 W}{nI_{ds}}\left[\frac{(V_{gst} - nV_{dssat})^3}{3} - \frac{(V_{gst})^3}{3}\right] = \frac{\mu (C'_{ox})^2 W}{nI_{ds}}\frac{(V_{gst})^3}{3} \qquad (\ 14\ )$$

In saturation, $I_{ds} = I_{dssat}$. Substituting (17) into (19), we get:

$$\frac{Q}{W} = \frac{\mu (C'_{ox})^2 W}{nI_{ds}}\frac{(V_{gst})^3}{3} = \frac{2}{3}LC'_{ox}V_{gst} \qquad (\ 15\ )$$

Therefore it is:

$$\langle Q' \rangle = \frac{1}{L}\int_0^L Q'(x)dx = \frac{2}{3}C'_{ox}V_{gst} \qquad (\ 16\ )$$

Example voltage amplifier

Let us consider the voltage amplifier as in lecture 7. We consider the variant with the MOSFET amplifier and resistor R_f. We use resistor R_f for DC feedback.

In Lecture 7 we derived the transfer function

$$V_o(s) = -V_s(s)\frac{C_i}{C_f}\alpha\frac{1}{(sT_r+1)}\frac{sT_f}{(sT_f+1)} \qquad (17)$$

The time constants are:

$$T_r = \frac{sC'_o\alpha}{\beta g_m} = \frac{s(C_iC_o+C_iC_f+C_fC_o)}{C_f g_m} \qquad (18)$$

with

$$C'_o = C_o + \frac{C_i^+C_f}{C_i^++C_f},\ \beta = \frac{C_f}{C_i^++C_f}, \alpha \equiv \frac{\beta\times g_m R_0}{1+\beta\times g_m R_0}, C_i^+ = C_i + C_g$$

And

$$T_f = R_f C_f \qquad (19)$$

If we have a voltage step with amplitude 1 at the input, the output voltage magnitude increases within 3 × T_r to about C_i/C_f and decreases again to 0 (Figure 3 above). (Gain is negative.)

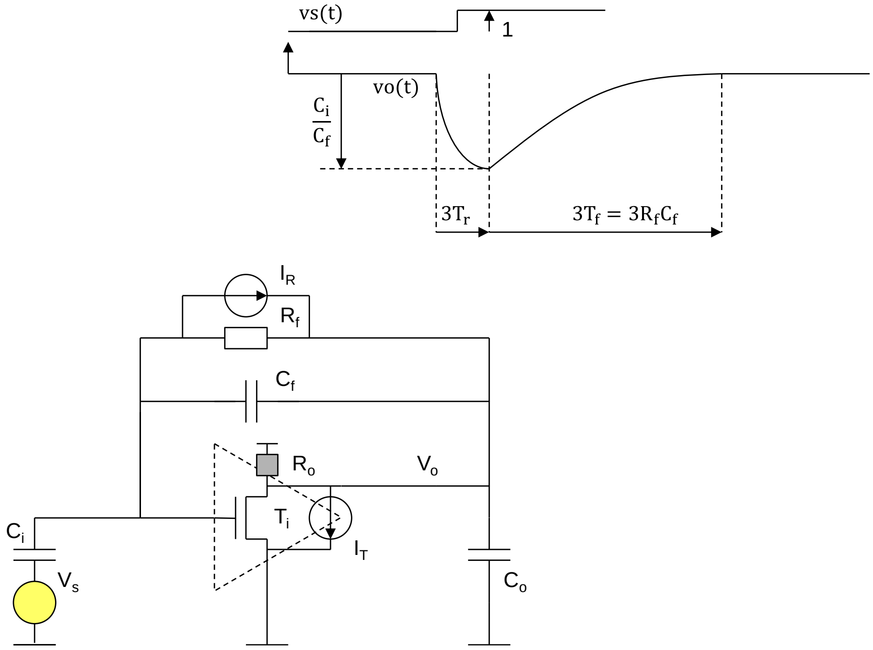


Figure 3: Voltage amplifier with noise sources. The figure above shows the step response.

The most significant noise sources in this circuit are the transistor T_i, represented by the noise current source I_T, and the feedback resistor Rf, represented by the noise current source I_R.

Components connected to the amplifier input generate substantial noise at the output because their noise is amplified by the circuit.

In contrast, the resistor R_o contributes relatively little to the total output noise since it is connected directly to the amplifier output, and its noise is not subject to amplification.

Figure 3 shows the circuit with the noise sources I_T and I_R (highlighted in white) and the input signal source Vs (highlighted in yellow). For the purpose of the noise analysis, Vs will be switched off.

We will now proceed with a frequency-dependent noise analysis of the circuit.

Step 1

We can move the transistor noise source to the input of the amplifier.

The idea behind this is as follows: the transistor noise can be modeled either as a current source IT between the drain and source, or as a voltage source V_T at the gate.

An additional voltage vg applied at the gate produces a drain-source current of $g_m \times v_g$, where gm is the transconductance. Since noise power densities are proportional to the square of the voltage or current, we have:

$$S_{VT} = \frac{S_{IT}}{g_m^2} = \frac{4\frac{2}{3}kTn}{g_m} \qquad (20)$$

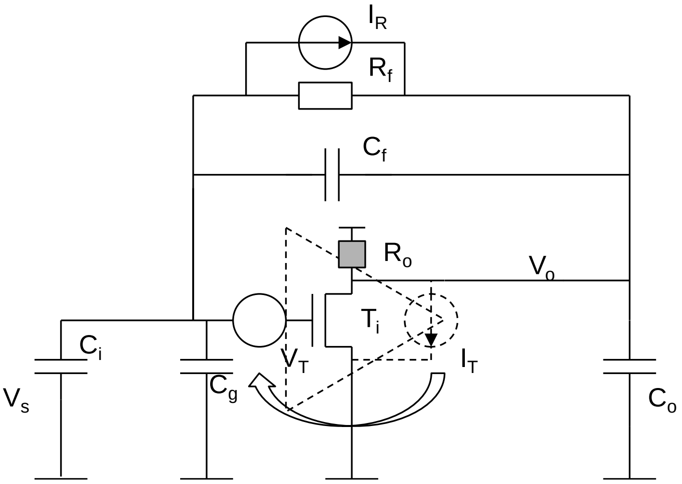
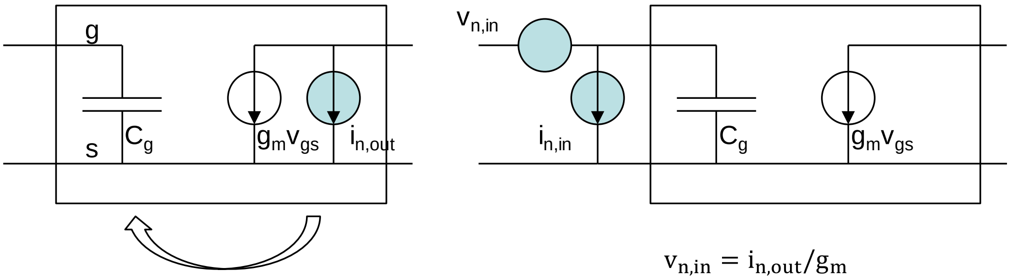


Figure 4: The noise source I_T can be moved to transistor input

Let us note the following:

If we move the noise sources to the input of a two-port circuit, and if that circuit has a finite input impedance, we must use a voltage source in series and a current source in parallel with the input.

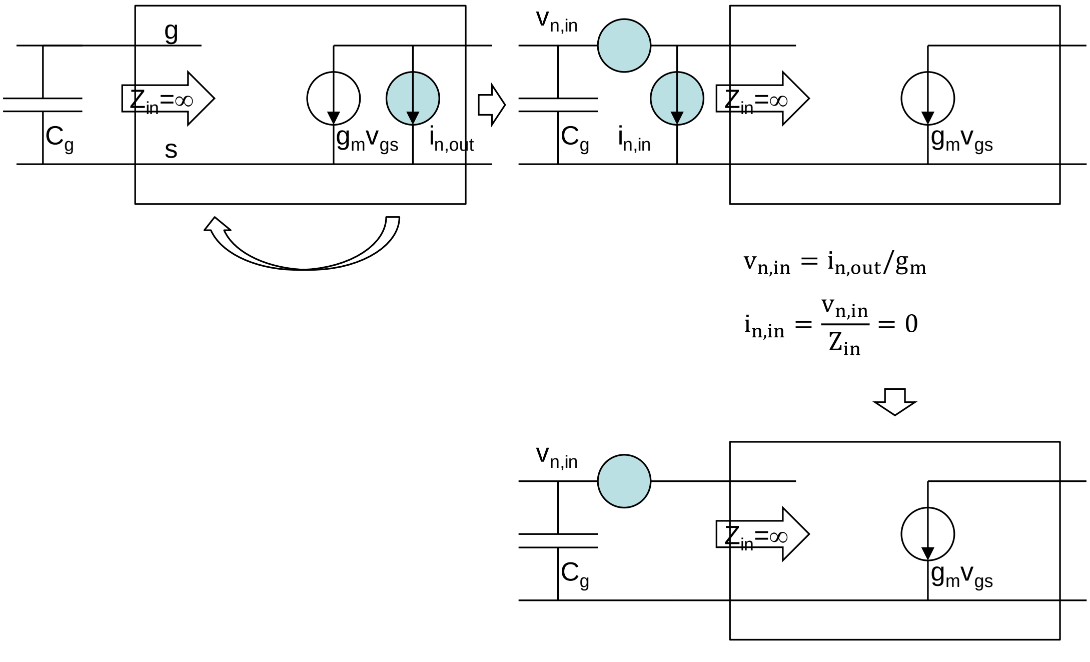


$$v_{n,in} = i_{n,out}/g_m$$

$$i_{n,in} = \frac{v_{n,in}}{Z_{in}} = sC_g v_{n,in}$$

Figure 5: Moving of the noise source - C_g inside two-port

However, we will define the gate and source, as the input of the transistor for noise analysis, after the C_{gs}. The two-port then has an infinite input impedance and the parallel current noise source has zero amplitude. A voltage source is sufficient for modelling the noise.



$$v_{n,in} = i_{n,out}/g_m$$

$$i_{n,in} = \frac{v_{n,in}}{Z_{in}} = 0$$

$$v_{n,in} = i_{n,out}/g_m$$

Figure 6: Moving of noise source - C_g in front of the two-port

Step 2: Moving of the Source I_R

The current source I_R is connected between the input and the output of the amplifier (Figure 3). The portion of the noise current that flows directly into the output contributes very little to the total output noise. In contrast, the portion of the noise current that flows into the input is **amplified** by the amplifier, producing a much larger noise contribution at the output.

It is therefore equal for the noise calculation whether the source is connected between the input and the output of the amplifier or between the input of the amplifier and ground.

We will connect the source I_R between the amplifier-input and the ground as this makes it easier to calculate the transfer function.

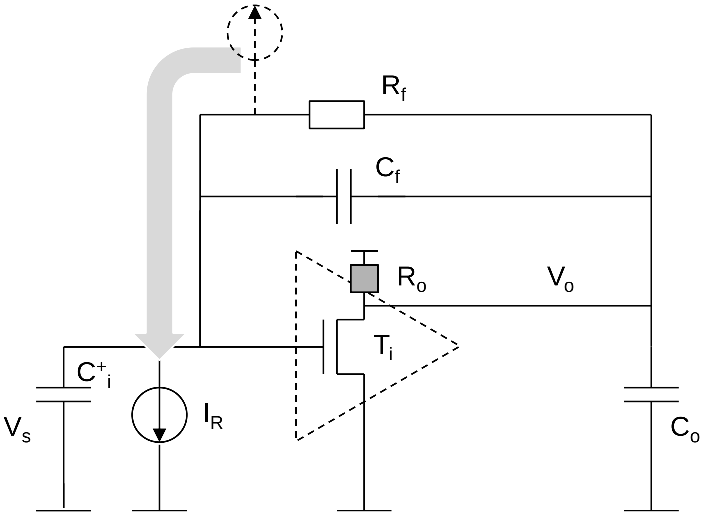


Figure 7: Moving of the noise source I_R

Step 3

To calculate the variance (noise power) of the noise signal at the amplifier output, we use the frequency-domain noise analysis. When multiple independent noise sources are present, their contributions can be evaluated separately. The total output noise power is obtained by summing the individual contributions:

$$< v_o^2 > = \int_0^\infty (S_{IR}(f)|H_{IR}(i\omega)|^2 + S_{VT}(f)|H_{VT}(i\omega)|^2)df \quad (21)$$

We therefore need to derive the transfer functions H(iω) corresponding to the two noise sources I_R and V_T.

Step 4

Transfer function for the I_R source.

We will perform a simplified analysis and assume that we have virtual ground at point v_i (Figure 8).

$$H(i\omega) = \frac{dV_o}{dI_R} = -Z_f = -\frac{R_f}{1+sR_fC_f} \quad (22)$$

The contribution of the noise source I_R to the total output noise variance is

$$\langle v_{oIR}\rangle^2 = \int_0^\infty S_{IR}\left|\frac{R_f}{1+i\omega T_f}\right|^2 df$$

Changing the integration variable from f to ω=2πf, we obtain

$$\langle v_{oIR}\rangle^2 = S_{IR}\frac{{R_f}^2}{2\pi T_f}\int_0^\infty \left|\frac{1}{1+i\omega T_f}\right|^2 d(\omega T_f)$$

Using $\int_0^\infty \left|\frac{1}{1+i\omega T_f}\right|^2 d(\omega T_f) = \arctan(\omega T_f)|_0^\infty = \frac{\pi}{2}$

we obtain: $\langle v_{oIR}\rangle^2 = S_{IR}\frac{{R_f}^2}{2\pi T_f}\frac{\pi}{2} = \frac{1}{4}\frac{1}{{C_f}^2}S_{IR}T_f$ (28)

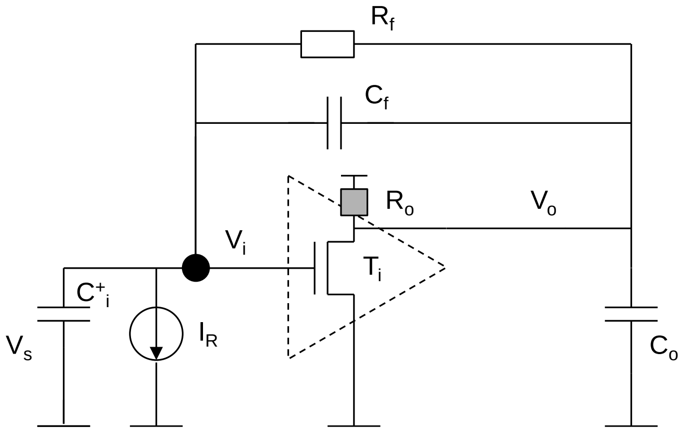


Figure 8: Noise source I_R

Interpretation

This result is easy to remember:

the noise power is proportional to the **noise spectral density** S multiplied by the **time constant** T_f.

Using $T_f = C_f R_f$ and $S_{IR} = 4kT/R_f$

the expression simplifies to

$\langle v_{oIR} \rangle^2 = \frac{kT}{C_f}$ (23)

Figure 9 shows the Bode diagram of the transfer function $|H|^2$. The yellow area contributes significantly to the integral in (28).

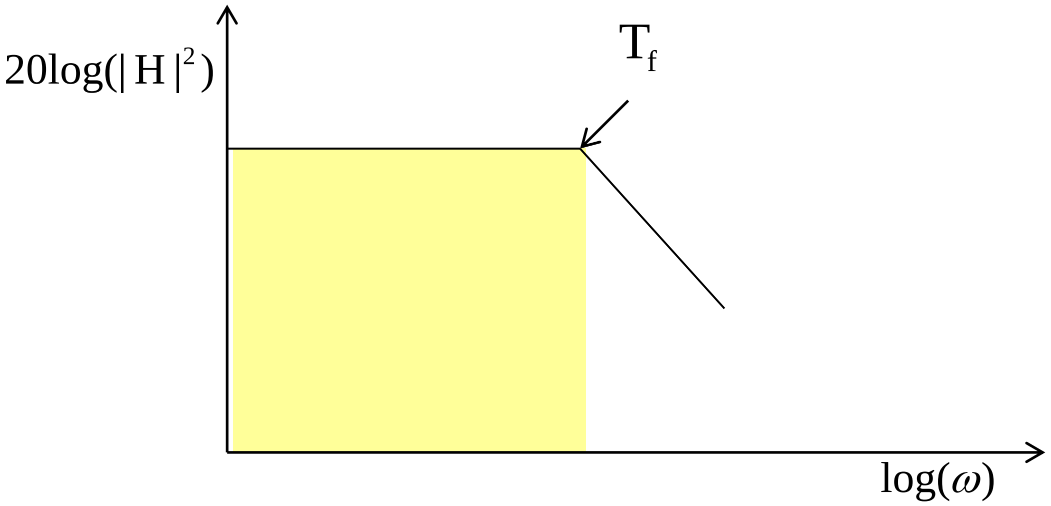


Figure 9: Bode plot of the H_{IR} transfer function

Step 5: Transfer Function for the Voltage Noise Source V_T

For the voltage noise source V_T, we derive the transfer function as follows. We assume that the gate voltage v_g is at virtual ground.

The output voltage is then given by

$$V_o = -\frac{Z_f + Z_i}{Z_i} V_T$$

Where

$Z_f = \frac{R_f}{1+i\omega C_f R_f}$ and $Z_i = \frac{1}{i\omega C_i^+}$

Substituting these expressions yields:

$$V_o = -\frac{\frac{R_f}{1 + i\omega C_f R_f} + \frac{1}{i\omega C_i^+}}{\frac{1}{i\omega C_i^+}} V_T$$

which simplifies to: $\frac{1+i\omega R_f(C_i^+ + C_f)}{1+i\omega R_f C_f} V_T$

The transfer function from the voltage noise source V_T to the output voltage is therefore

$$H(i\omega) = \frac{V_o}{V_T} = \frac{1+i\omega R_f(C_i^+ + C_f)}{1+i\omega R_f C_f} = \frac{1+i\omega T_z}{1+i\omega T_f};\ T_Z = R_f(C_i^+ + C_f)\ ;\ T_f = R_f C_f \qquad (31)$$

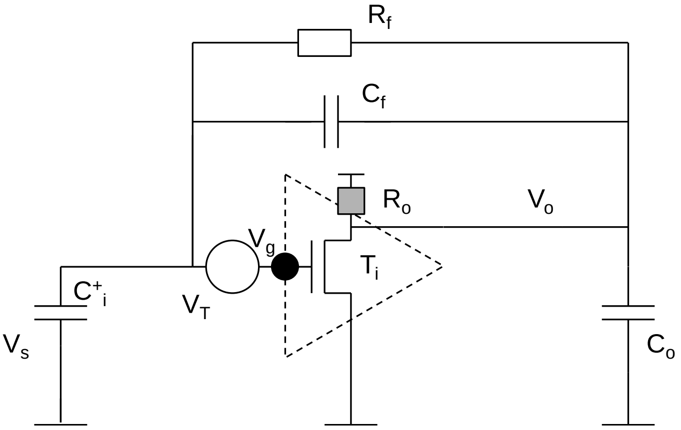


Figure 10: Noise source V_T

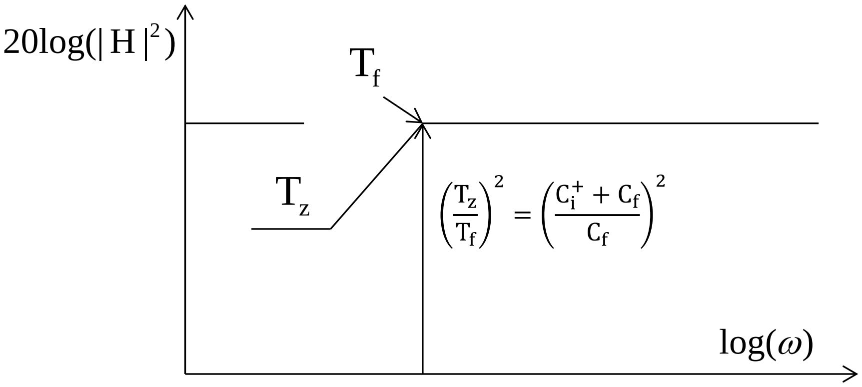


Figure 11: Bode plot of the H_{IR} transfer function, case without Tr

Note that this transfer function predicts constant gain for high frequencies.

If we were to substitute this transfer function directly into the noise power integral, the result would be **infinite noise power**. This is clearly unphysical and indicates that the model is incomplete.

At high frequencies, the **virtual ground assumption** is no longer valid. In Lecture 7, we performed a more detailed analysis of the circuit and showed that the transfer function contains **two poles**, corresponding to two distinct time constants.

$$T_r = \frac{sC'_o\alpha}{\beta g_m} \qquad (24)$$

$$T_f = R_f C_f \qquad (25)$$

The circuit in Lecture 7 is the same as the circuit in Figure 10 except for the position of the input source. The position of the input source does not change the time constants in the denominator of the transfer function (the poles).

We can extend the transfer function by including the second (high-frequency) time constant:

$$H(i\omega) = \frac{V_o}{V_T} = \frac{(1+i\omega T_Z)}{(1+i\omega T_r)(1+i\omega T_f)} \qquad (26)$$

The time constants are given by: $T_r = \frac{sC'_o}{\alpha\beta g_m}$; $T_f = R_f C_f$; $T_z = R_f(C_f + C_i^+)$

The time constants satisfy the ordering: $T_z > T_f > T_r$

Including the additional pole T_r ensures that the gain rolls off at high frequencies, resulting in a **finite output noise power**, as required by physical reality.

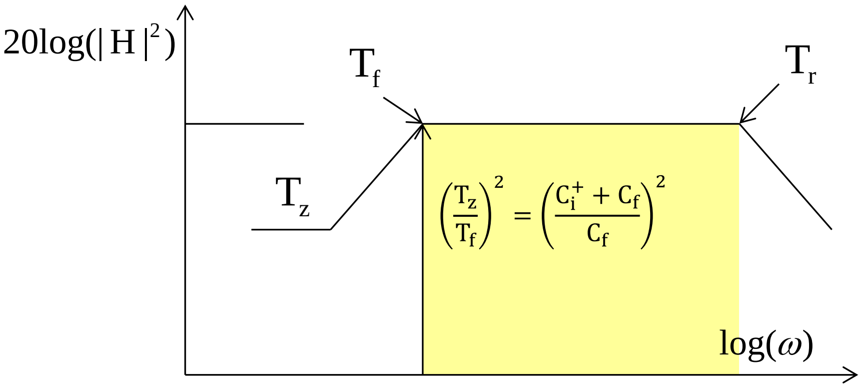


Figure 12: Bode plot of transfer function H_{IR}, case with T_r

The frequency range between $1/T_f$ and $1/T_r$ (highlighted in yellow) provides the **dominant contribution** to the noise integral.

$$H(i\omega) \sim \frac{T_z}{T_f(1+i\omega T_r)} = \frac{C_f + C_i^+}{C_f}\frac{1}{1+i\omega T_r} \qquad (27)$$

The output noise variance due to the voltage noise source V_T can be approximated as:

$$< v_{oT} >^2 = \int_0^\infty S_{VT}\left(\frac{C_f + C_i^+}{C_f}\right)^2 \left|\frac{1}{1+i\omega T_r}\right|^2 df = S_{VT}\left(\frac{C_f + C_i^+}{C_f}\right)^2 \frac{1}{2\pi T_r}\frac{\pi}{2} = S_{VT}\left(\frac{C_f + C_i^+}{C_f}\right)^2 \frac{1}{4T_r} \qquad (36)$$

with $S_{VT} = \frac{4kTn2/3}{g_m}$ und $T_r = \frac{sC\prime_o}{\alpha\beta g_m}$

Summary

The variance of the noise signal at the output of the voltage amplifier is given by the following formulas:

$$< v_o >^2 = < v_{oR} >^2 + < v_{oT} >^2 = \frac{1}{4}\frac{1}{C_f^2}((C_i^{+} + C_f)^2 \frac{S_{VT}}{T_r} + S_{IR}T_f) \qquad (28)$$

$$S_{IR} = \frac{4kT}{R_f} \qquad (29)$$

$$S_{VT} = \frac{4kTn2/3}{g_m} \qquad (30)$$

Signal-to-noise ratio (SNR)

We now calculate the **signal-to-noise ratio (SNR)**

A voltage step at the input with amplitude $V_{i,sig}$ produces an output signal with amplitude: (Figure 13)

$$V_{osig}^2 = V_{isig}^2 \left(\alpha \frac{C_i}{C_f}\right)^2 \sim V_{isig}^2 \left(\frac{C_i}{C_f}\right)^2 \quad (31)$$

The inverse squared SNR is therefore

$$\frac{<v_o>^2}{V_{osig}^2} = \frac{1}{SNR^2} = \frac{1}{V_{osig}^2} \frac{1}{4} \frac{1}{C_f^2} \left(\left(C_i^+ + C_f\right)^2 \frac{S_{VT}}{T_r} + S_{IR} T_f\right) \quad (32)$$

or

$$\frac{<v_o>^2}{V_{osig}^2} = \frac{1}{SNR^2} = \frac{1}{V_{isig}^2} \frac{1}{4} \frac{1}{C_i^2} \left(\left(C_i^+ + C_f\right)^2 \frac{S_{VT}}{T_r} + S_{IR} T_f\right) \quad (33)$$

Equivalent Noise Signal (Input-Referred Noise)

An **equivalent noise signal**—also called the **equivalent noise voltage (ENV)**—is defined as the input signal amplitude that yields SNR=1.

$$ENV^2 = \frac{1}{4} \frac{1}{C_i^2} \left(\left(C_i^+ + C_f\right)^2 \frac{S_{VT}}{T_r} + S_{IR} T_f\right)$$

This equivalent noise signal is equal to the **input-referred noise** of the amplifier.

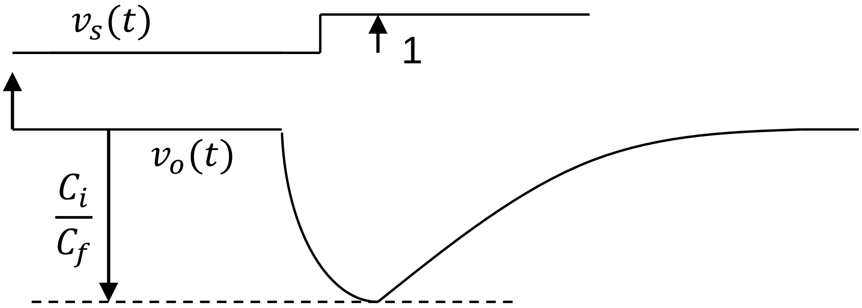


Figure 13: Step response

Optimization for Maximum SNR

We now aim to optimize the amplifier parameters in order to achieve **maximum signal-to-noise ratio (SNR)**.

Which parameters can be varied?
During optimization, we require the **signal gain** C_i/C_f to remain constant. This can be achieved by increasing C_i and C_f by the same scaling factor.

Effect on Resistor Noise Contribution

Increasing both C_i and C_f reduces the contribution of the feedback resistor noise. The equivalent input-referred noise due to R_f is

$$\mathrm{ENS}_R^2 = \frac{1}{4} S_{IR} \frac{T_f}{C_i^2} = \frac{1}{4A^2}\frac{R_f}{C_f}\frac{4kT}{R_f} = \frac{kT}{A^2 C_f} \quad (\ 34\ )$$

Where $A = C_i / C_f$

Thus, increasing C_f (while keeping $A = C_i / C_f$ constant) **reduces the resistor noise contribution**.

Effect on Transistor Noise Contribution

The transistor noise contribution remains unchanged under this scaling:

$$\mathrm{ENS}_T^2 = \frac{kT2/3}{g_m}\left(\frac{C_f + C_i^+}{C_i}\right)^2 \frac{1}{T_r} \qquad (\ 35\ )$$

Effect of Output Capacitance C_o

We can also increase the output capacitance C_o. A larger C_o makes the amplifier **slower**, which helps to reduce the transistor noise contribution. This follows from the expression for the high-frequency time constant:

$$T_r = \frac{sC\prime_o \alpha}{\beta g_m} \qquad (\ 36\ )$$

Increasing C_o' increases T_r, thereby reducing ENS_T^2.

Effect of Transconductance g_m

Finally, we can increase the transconductance g_m by scaling the transistor size and the load resistor R_o.

For a constant output capacitance C_o, a larger g_m makes the amplifier faster. However, since g_m cancels out in the expression for ENS_T^2, the transistor noise contribution remains essentially unchanged.

Optimizing the Amplifier for Required SNR, Gain, and Bandwidth (Summary)

We begin with a **minimum-size design**, choosing a small feedback capacitor C_f (e.g., 10 fF), a small transistor, and a low bias current (e.g., 10 µA).

Reduce feedback resistor noise
Scale up C_f and C_i by the same factor until the noise contribution of the feedback resistor R_f becomes sufficiently small, while keeping the gain C_i/C_f constant.

Reduce transistor noise
Increase the output capacitance C_o until the transistor noise contribution is reduced to the required level.

Verify bandwidth
Check whether the amplifier meets the required bandwidth, i.e., whether it is sufficiently fast.

Restore speed if necessary
If the amplifier is too slow, increase the transconductance gm by scaling the transistor size and the load resistor R_o.

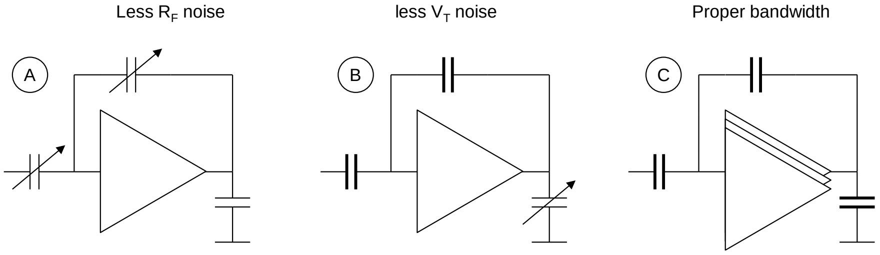


Figure 14: Optimization method

The following applies to most analogue circuits (see e.g. formula 37):

$SNR^2 \sim 1/T_r$

$SNR^2 \sim 1/g_m$ = 1/ power consumption

Sometimes, the following figure of merit is defined as a measure of how well a circuit has been optimized:

FOM = power consumption Tr $/SNR^2$

This figure of merit should be similar for all well optimized circuits.

Circuit with smaller FOM is better. The figure of merit allows us to compare different circuits.

Additional examples (not for exam)

Charge sensitive amplifier

A **charge-sensitive amplifier (CSA)** is an important circuit. The input signal is a current pulse carrying a total charge QQ; the output voltage is proportional to QQ and independent of the detailed shape of the pulse.

A charge-sensitive amplifier (Figure 15) has the same basic structure as a voltage amplifier.

Let us now derive the transfer function of the charge-sensitive amplifier. We begin with the transfer function of the voltage amplifier.

The transfer function of the voltage amplifier is:

$$V_o(s) = -V_s(s)\frac{C_i}{C_f}\alpha\frac{1}{(sT_r+1)}\frac{sT_f}{(sT_f+1)} \qquad (37)$$

The input voltage source (Figure 3) can be converted into an equivalent current source (Figure 15). The following relationship holds between the current of the equivalent current source and the voltage of the original voltage source:

$$\frac{i_s(s)}{sC_i} = v_s(s) \quad (45b)$$

In the time domain, the current i(t) is equal to the time derivative of the voltage v(t) multiplied by the capacitance C_i:

If we substitute (45b) into (22), we obtain the transfer function of the charge amplifier:

$$V_o(s) = -\frac{i_s(s)}{sC_f}\alpha\frac{sT_f}{(sT_r+1)(sT_f+1)} = -\frac{Q}{C_f}\alpha\frac{1}{(sT_r+1)}\frac{sT_f}{(sT_f+1)} \qquad (45c)$$

From this expression, it follows that a current pulse with total charge Q produces an output signal whose peak amplitude is:

$$V_{osig} \sim \frac{Q_{isig}}{C_f} \quad (38)$$

The output voltage increases to the maximum amplitude (Q/C_f) within $3 \times T_r$ and goes back down to 0. (The gain is negative.)

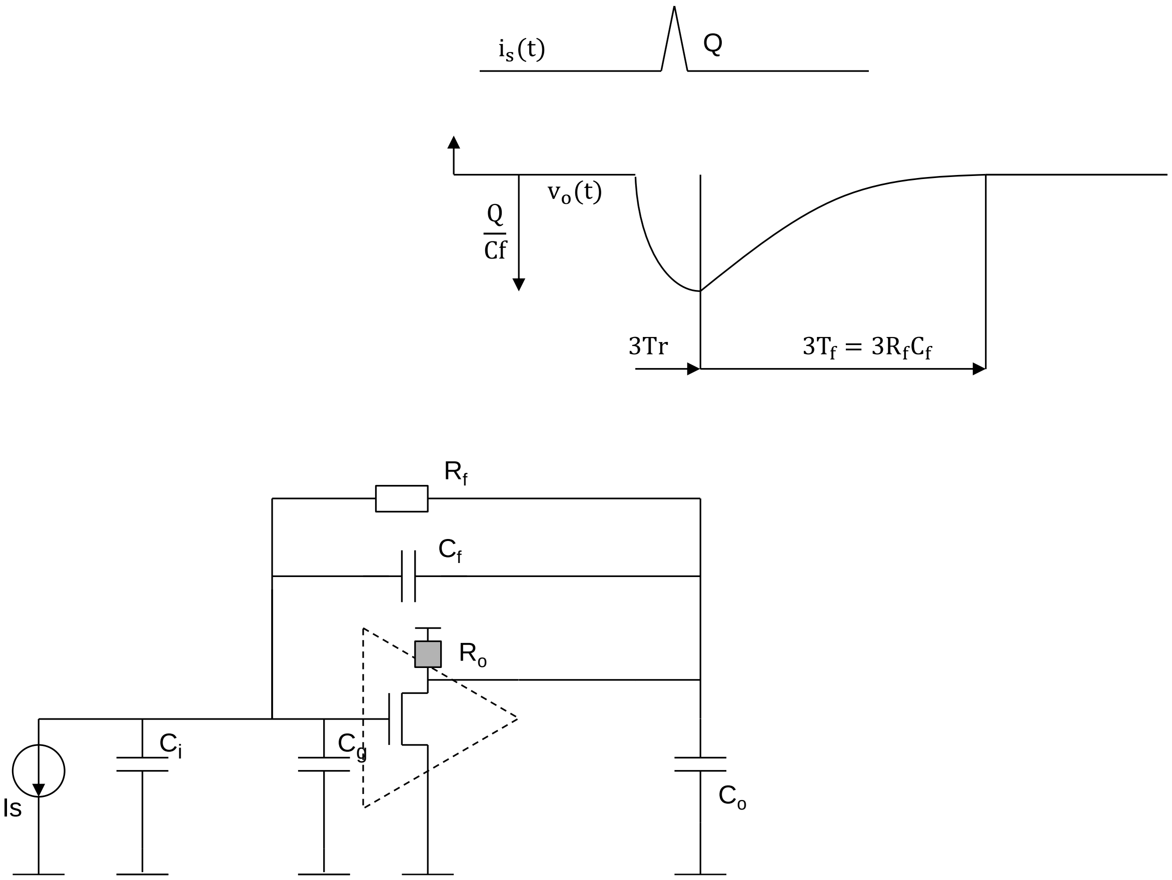


Figure 15: Charge sensitive amplifier. Above - pulse response

There are three main sources of noise in the circuit (Figure 16): the input transistor (source V_T), the resistor R_f (source I_R).

The third noise source appears when a sensor with a leakage current is connected to the amplifier as the signal source; this leakage current produces an additional noise component I_D.

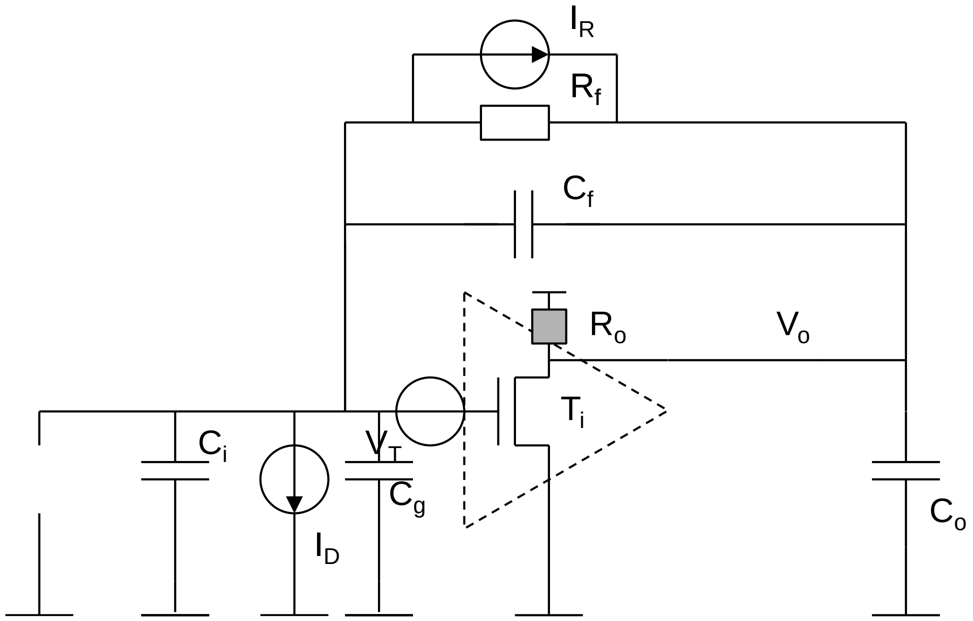


Figure 16: Charge sensitive amplifier. Sources of noise

Since the circuit has the same form as the voltage amplifier, the same formulas also apply.

Therefore, the variance of the noise signal at the output is:

$$< v_o >^2 = \frac{1}{4}\frac{1}{C_f^2} \left((C_i^{+} + C_f)^2 \frac{S_{VT}}{T_r} + S_{IR}T_f + S_{ID}T_f\right) \qquad (39)$$

The power spectral densities are

$$S_{VT} = \frac{4kTn2/3}{g_m}$$

(thermal noise)

$$S_{IR} = \frac{4kT}{R_f}$$

(thermal noise)

$$S_{ID} = 2eI_{leak}$$

(leakage current noise)

Inverse signal to noise ratio (SNR) is:

$$\frac{<v_o>^2}{V_{osig}^2} = \frac{1}{SNR^2} = \frac{1}{V_{osig}^2}\frac{1}{4}\frac{1}{C_f^2}\left((C_i^{+} + C_f)^2 \frac{S_{VT}}{T_r} + S_{IR}T_f\right) \qquad (40)$$

with

$$V_{osig}^{\square} = \frac{Q_{isig}}{C_f}$$

Therefore:

$$\frac{<v_o>^2}{V_{osig}^2} = \frac{1}{SNR^2} = \frac{1}{Q_{isig}^2}\frac{1}{4}\left((C_i^{+} + C_f)^2 \frac{S_{VT}}{T_r} + S_{IR}T_f\right) \qquad (41)$$

The equivalent noise charge (ENC) is defined as the input charge that results in a signal-to-noise ratio SNR=1. In other words, it is the amount of input charge whose output signal amplitude equals the rms noise amplitude.

The ENC is given by:

$$ENC^2 = \frac{1}{4}((C_i^{+} + C_f)^2 \frac{S_{VT}}{T_r} + S_{IR}T_f + S_{ID}T_f) \qquad (42)$$

An input signal equal to the ENC produces an output signal whose amplitude is equal to the standard deviation of the noise. It therefore represents approximately the smallest measurable signal when the measurement is performed only once.

To maximize the signal-to-noise ratio (i.e., to minimize the ENC), the parameters of the charge amplifier must be optimized.

It is clearly advantageous to minimize the input capacitance (primarily the sensor capacitance). This reduces the noise contribution of the input transistor.

Choosing a small feedback capacitance Cf reduces the contribution of the feedback resistor noise. The noise contribution from the feedback resistor can be written as:

$$ENC_R^2 = \frac{1}{4} S_{IR}R_fC_f$$

The capacitance C_o can be increased to reduce the transistor noise contribution. A larger C_o slows down the amplifier (the rise time Tr increases), which lowers the effect of the transistor voltage noise:

$$T_r = \frac{sC'_o}{\alpha\beta g_m}$$

The transconductance g_m can be increased by using multiple transistors in parallel and increasing R_o. For a constant C_o, a larger g_m makes the amplifier faster while leaving the noise contribution unchanged.

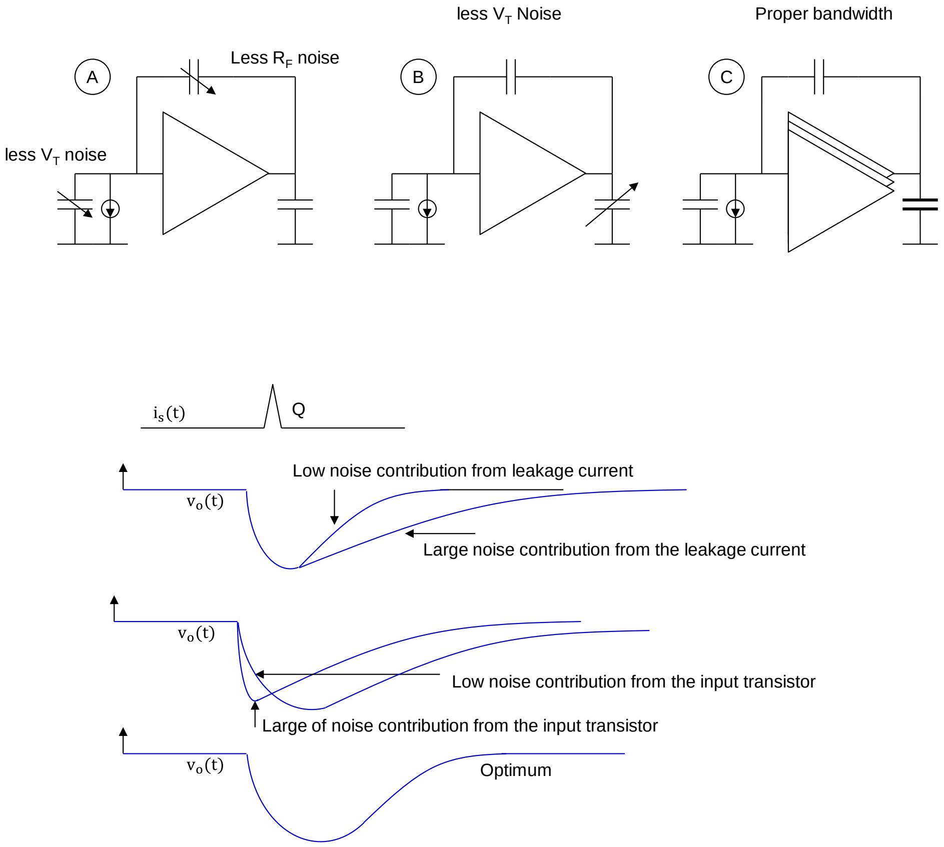


Figure 17: Noise and time constants

Figure 17 illustrates the optimization of the charge-sensitive amplifier for maximum SNR.

It is important to minimize the input capacitance C_i.
A short time constant T_f minimizes the noise contribution from the sensor leakage current.
A slow rise time constant T_r helps reduce the noise contribution of the input transistor.

It can be shown that the signal-to-noise ratio (SNR) is maximized when the two time constants are of comparable magnitude: $T_f \sim T_r$

1/f noise and transistor mismatch

We will consider two effects that, at first glance, may appear quite different.

If measurements are performed on a group of transistors with identical layouts, it is observed that the drain currents are slightly different even when all other parameters—such as bias voltages and temperature—are the same. This inequality of currents is referred to as mismatch. Mismatch is time-independent and therefore represents a permanent deviation. It is statistically described by its variance. The actual drain current of a given transistor is a random variable characterized by a mean value and a variance.

1/f noise (also called *flicker noise*) refers to temporal variations of the transistor drain current.

Although mismatch and 1/f noise seem different, both phenomena can be described as fluctuations of charge in the transistor channel. These charge fluctuations originate from so-called trap states in the gate oxide near the silicon–oxide interface.

In the gate oxide near the silicon interface, there exist trap states (often simply called *traps*). These traps can capture charge carriers from the transistor channel.

The mean number of charge carriers captured during a time interval Δt is given by:

$$< \Delta N > \ = N_{trap} c_r \ \Delta t$$

Here, c_r denotes the capture rate. The product $N_{trap}c_r\Delta t$ represents the average number of charge carriers captured by the traps during the time interval Δt.

These charge carriers are released with a time constant (average time) τ

The captured charge carriers are released with a characteristic time constant (mean lifetime) τ.

As a result, the excess number of trapped charge carriers decays exponentially with time:

$$\Delta N(t) = \Delta N e^{-\frac{t}{\tau}}$$

Here, c_r denotes the capture rate, N_{trap} is the number of available traps, and ΔN is the number of charge carriers captured during the time interval Δt.

The mean number of charge carriers captured in the time interval Δt is given by:

$$< \Delta N >= N_{trap} c_r \ \Delta t$$

In strong inversion, the transistor drain current depends on the total charge stored in the channel. In saturation, this relationship can be written as:

$I_{dssat} = \frac{3}{4n}\frac{\mu}{L^2} V_{gst} Q = kQ$ where Q is the total channel charge and k is a proportionality constant.

When electrons are trapped, the amount of charge in the channel is reduced. As a consequence, the drain current decreases as shown in Figure 18.

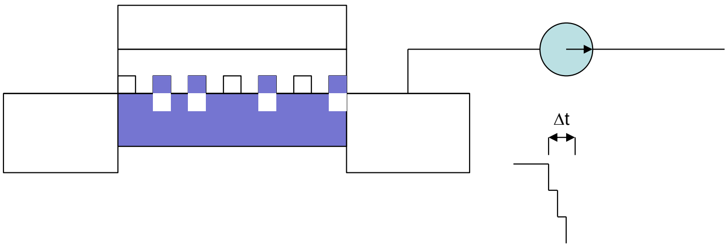


Figure 18: Charge carriers are trapped - current drops

Since the trapped electrons are released again with a characteristic time constant τ, the channel charge—and therefore the drain current—recovers exponentially, as shown in Figure 19. The resulting current fluctuation is given by:

$$dI_{dssat} = ke\Delta N e^{-\frac{t}{\tau}}$$

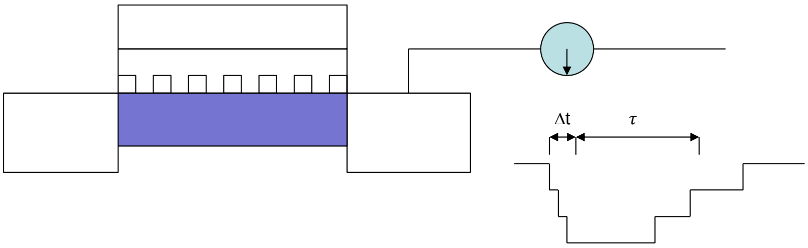


Figure 19: Charge carriers are released – current increases

Since the process of trapping and releasing electrons is repeated continuously, it gives rise to **current noise** (Figure 20).

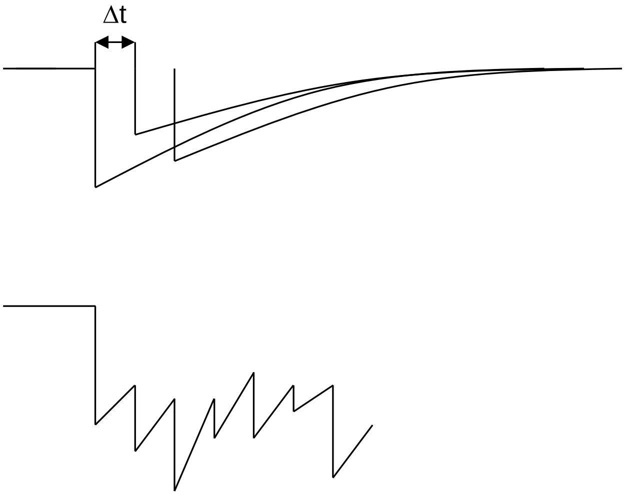


Figure 20: Trapping and releasing electrons creates noise

We now ask: *what is the resulting power spectral density S(f)?*

Capturing charge carriers during a time interval Δt produces a current pulse of the form:

$$dI_{dssat} = ke\Delta N e^{-\frac{t}{\tau}}$$

Here, eΔN is the trapped channel charge, which we denote by:

$Q = e\Delta N$

We therefore define the **noise input signal** as the trapped channel charge Q, and the corresponding **noise input pulse** as:

$$P_{in}(t) = ke^{-\frac{t}{\tau}}$$

According to the noise theory introduced in Lecture 10, the power spectral density of the resulting noise is given by:

$S(f) = 2\frac{<Q^2>}{\Delta t}|\,P_{in}(i\omega)|^2$

We assume that the number of trapped charge carriers follows **Poisson statistics**. Therefore,

$\langle Q^2\rangle = e^2\langle \Delta N^2\rangle = e^2\langle \Delta N\rangle$

Using $\langle \Delta N\rangle = c_r\langle N_{trap,\tau}\rangle\Delta t$

we obtain: $\langle Q^2\rangle = e^2\langle N_{trap,\tau}\rangle\Delta t$

The noise input pulse is

$P_{in}(t) = k e^{-\frac{t}{\tau}}$

with Fourier transform:

$P_{in}(\omega) = \frac{k\tau}{1+i\omega\tau}$

Substituting into the expression for the spectral density yields:

$S_{I,\tau}(f) = 2\frac{<Q^2>}{\Delta t}|\,P_{in}(i\omega)|^2 = 2e^2c_r\langle N_{trap,\tau}\rangle\frac{\tau^2k^2}{1+\omega^2\tau^2}$

From the previous slide, the current noise power spectral density associated with traps having a single time constant τ is:

$S_{I,\tau}(f) = 2e^2c_r\langle N_{trap,\tau}\rangle\frac{\tau^2k^2}{1+\omega^2\tau^2}$

This expression describes only traps characterized by the time constant τ.

In practice, trap time constants are distributed over a wide range, from τ_{min} to τ_{max}

The total current noise power spectral density is obtained by integrating over all trap time constants:

$S_I(f) = \int_{\tau_{min}}^{\tau_{max}} S_{I,\tau}(f)$

Substituting the expression above yields:

$S_I(f) = 2e^2c_rk^2\int_{\tau_{min}}^{\tau_{max}}\langle N_{trap}\rangle\frac{\tau^2}{1+\omega^2\tau^2}d\tau$

For a broad and approximately uniform distribution of traps over logarithmic time constants, this integral evaluates to:

$S_I(f) \propto \frac{e^2\langle N_{trap}\rangle k^2}{\omega}$

Thus, the result exhibits the characteristic **1/f (flicker) noise behavior**.

The following relations are also used:

$k = \frac{3}{4n}\frac{\mu}{L^2}V_{gst}$; $I_{dssat} = \frac{1}{2n}\mu C'_{ox}\frac{W}{L}V_{gst}^2$; $\omega=2\pi f$

Combining these expressions, the drain current noise spectral density becomes:

$S_I(f) \propto \frac{\mu e^2\langle n_t\rangle I_{dssat}}{L^2nC'_{ox}f}$

Here, n_t denotes the **trap density per unit area**.

To explain **transistor mismatch**, we can assume that charge carriers remain permanently trapped. In this case, if we repeatedly measure the current of a single transistor, we always obtain the same value. However, if we measure the current across a group of transistors with identical layouts and operating conditions, the current of each transistor depends on the number of trapped charge carriers in that specific device. This number varies from transistor to transistor, leading to **mismatch** (Figure 21).

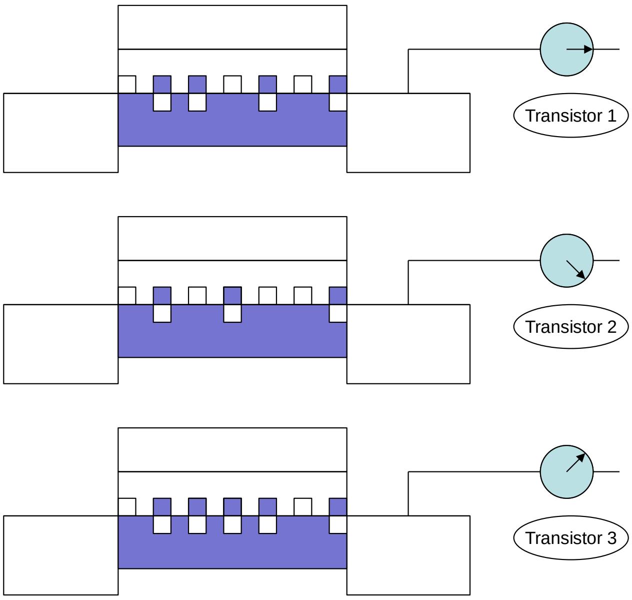


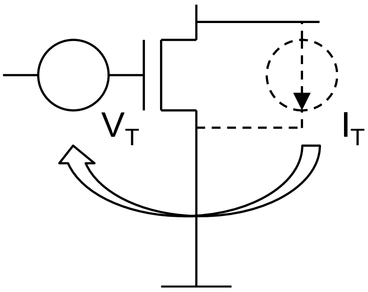
Figure 21: Mismatch. The number of trapped charge carriers varies from transistor to transistor

In both cases, the current variation originates from the trapped charge carriers. The difference is that **1/f noise** represents fluctuations over time in a single transistor, while **mismatch** represents fluctuations across a population of transistors. Since the underlying mechanism is the same, we can expect that the formula for the variance of the current is identical in both cases.

It holds

$$\mathrm{Var}(I_{ds}) \sim \int S df \sim \frac{\mu e^2 \langle n_t \rangle I_{dssat}}{L^2 n C'_{ox}}$$

We can also move the noise source to transistor gate.



The following then applies:

$$S_V = \frac{S_I}{g_m^2} \sim \frac{\mu e^2 \langle n_t \rangle I_{dssat}}{L^2 n C'_{ox} g_m^2 f}$$

or

$$S_V \sim \frac{\mu e^2 \langle n_t \rangle I_{dssat}}{L^2 n C'_{ox} g_m^2 f} \sim \frac{e^2 \langle n_t \rangle}{LWnC'^2_{ox}\, f}$$

We write it simplified:

$$S_V = \frac{k_{1/f}}{f} \text{ mit } k_{1/f} \sim \frac{e^2 \langle n_t \rangle}{LWnC'^2_{ox}}$$

Transistors with a large gate area have small 1/f noise.

From the book by B. Razavi (Design of Analog CMOS Integrated Circuits):

> “It is therefore not surprising to see devices having areas of several hundred square microns in low-noise applications.”
>
> “PMOS devices exhibit less 1/ f noise than NMOS transistors because the former carry the holes in a “buried channel,” i.e., at some distance from the oxide-silicon interface, and hence trap and release the carriers to a lesser extent.”

1/f noise output of the charge amplifier

Let us now calculate the contribution of the 1/f noise to the noise signal at the output of the charge sensitive amplifier. We can use the transfer function for the V_T.

$$H(i\omega) = \frac{V_o}{V_T} = \frac{(1+i\omega T_Z)}{(1+i\omega T_r)(1+i\omega T_f)} \qquad (43)$$

with:

$$T_r = \frac{C_iC_o+C_iC_f+C_fC_o}{C_f g_m}$$

$$T_f = R_fC_f$$

$$T_z = R_f(C_f + C_i^+)$$

1/f noise contribution is given by the integral:

$$< v_{o1/f\,T} >^2 = \int_0^\infty S_{V1/fT}|H(f)|^2\,df \quad (66)$$

The frequency range between $1/T_z$ and $1/T_f$ contributes the most to the integral. In this frequency range, we can simplify the formula (66):

$$< v_{o1/f\,T} >^2 = k_{1/f}\int_{1/T_z}^{1/T_f}\frac{\omega T_Z^2}{1+\omega^2T_f^2}\,d\omega = \frac{k_{1/f}}{2}\,\frac{T_Z^2}{T_f^2}\ln(1+\omega^2T_f^2)\,|^{1/T_f}{}_{1/T_z} \sim k_{1/f}\frac{(C_f+C_i^+)}{C_f^2}$$

Interestingly, the integral does not depend on time constants when the ratio T_z/T_f is constant! For smaller time constants, $S_{1/f}$ decreases but the frequency range in the integral ($1/T_z$ to $1/T_f$) increases.

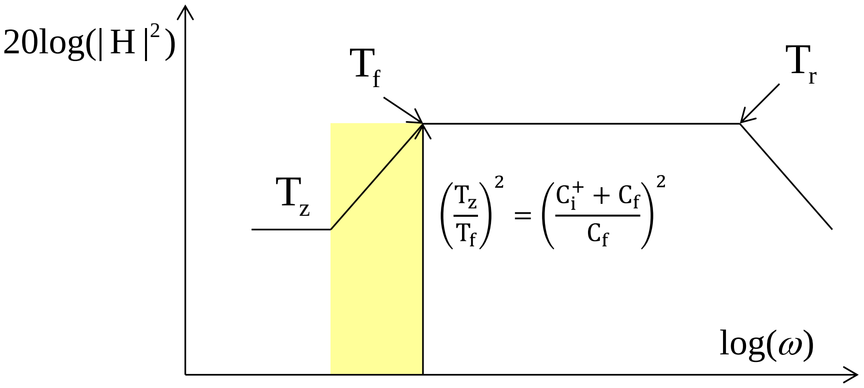


Figure 22: Frequency range between $1/T_z$ and $1/T_f$ contributes most to the integral

The variance of the noise signal at the output of the charge amplifier then becomes:

$$< v_o >^2 = \frac{1}{4}\frac{1}{C_f^2}\left((C_i^+ + C_f)^2\left(\frac{S_{VT}}{T_r} + k_{1/f}\right) + S_{IR}T_f + S_{ID}T_f\right)$$

The power spectral densities are:

$$S_{VT} = \frac{4kTn2/3}{g_m}$$

(thermal noise)

$$S_{IR} = \frac{4kT}{R_f}$$

(thermal noise)

$S_{ID} = 2eI_{leak}$

(leakage current noise)

$$k_{1/f} \sim \frac{\mu e^2 \langle n_t \rangle I_{dssat}}{L^2 n C'_{ox} g_m^2 f} \sim \frac{e^2 \langle n_t \rangle}{2LWnC'^2_{ox}}$$

(1/f noise)

When the detector leakage current is small and the time constant T_r is large, the 1/f noise dominates. W and L of the transistor should then be increased. A PMOS input transistor has smaller constant $k_{1/f}$.

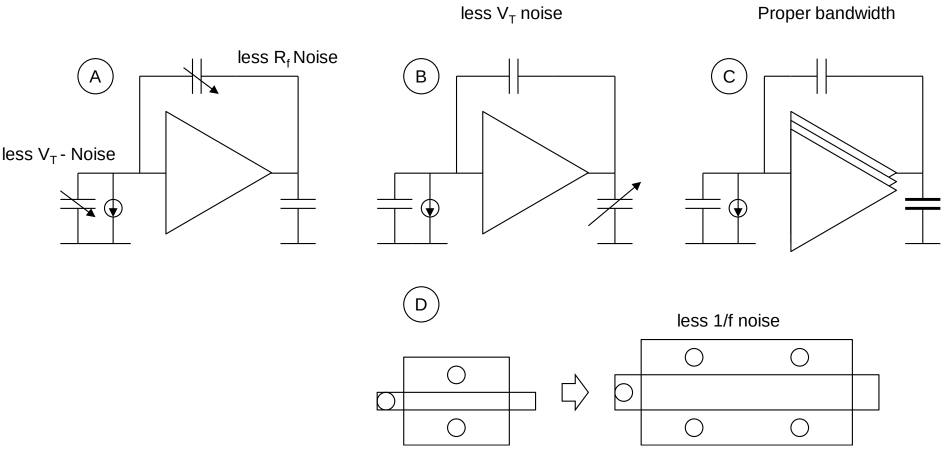


Figure 23: Optimization method

Summary

Figure 24, Figure 25 and Figure 26 illustrate different noise sources.

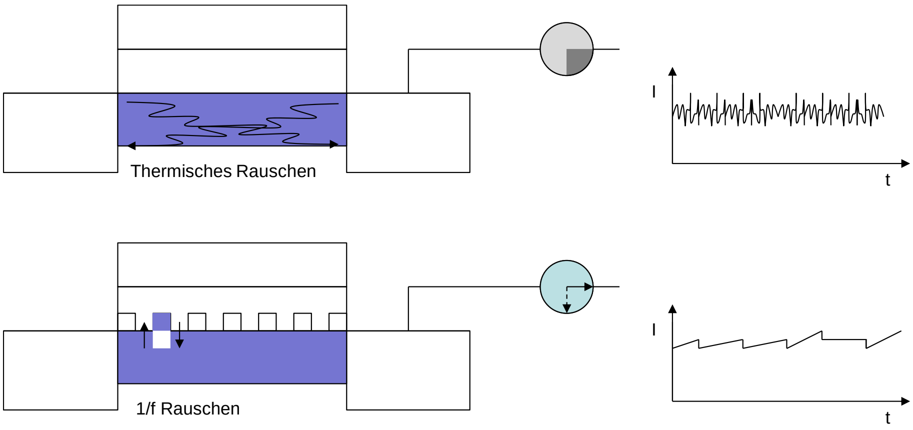


Figure 24: Noise in the transistor

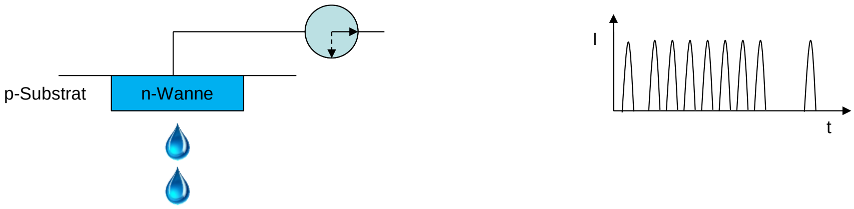


Figure 25: Sensor leakage current noise

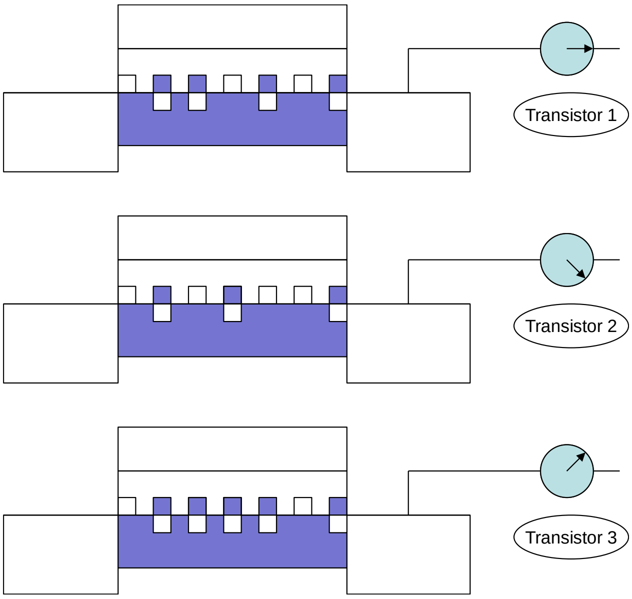


Figure 26: Transistor mismatch