Lecture 14

The topics of this lecture are:

- Noise Theory and Thermal Noise in Resistor
- Noise in the MOS transistor
- Noise in the voltage amplifier and the charge sensitive amplifier
- 1/f noise and transistor mismatch

Noise Theory Summary (see also DAS_2023_Noise_Theory)

This chapter is a summary of the old lecture: **DAS_2023_Noise_Theory**.

Electronic noise is our perception of fluctuations in current flow or voltage.

In order to calculate the amplitude of the noise signal, it is necessary to know and mathematically describe the source of the noise (the input variable) and the transfer function of the measurement device (the circuit attached to the noise source).

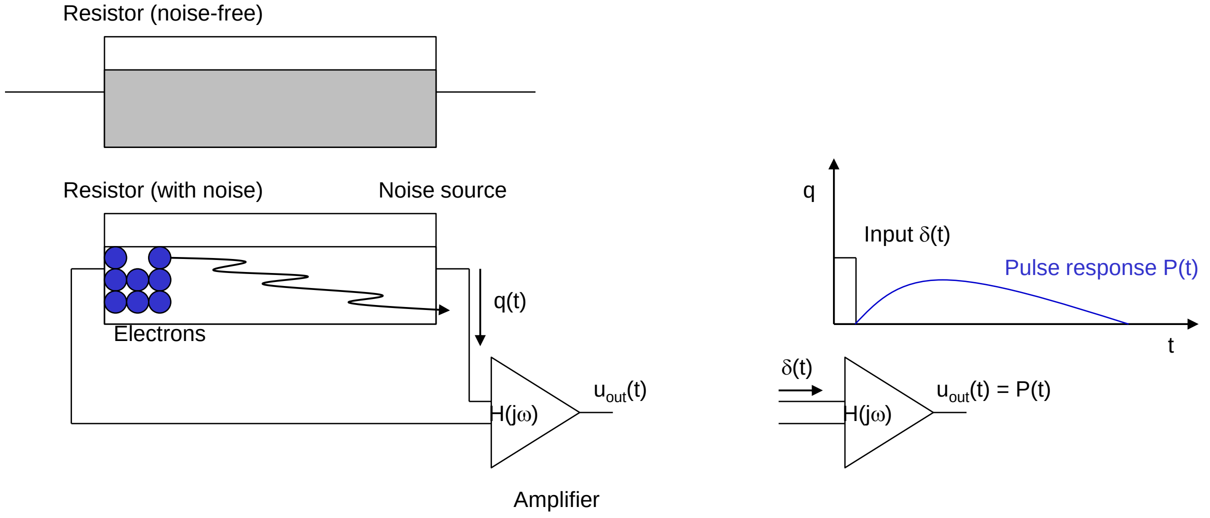
If the circuit remains unchanged in time, the transfer function can be represented as a frequency-dependent complex function H(s) (quotient of Fourier or Laplace-transformed signals).

$H(s) = V_{out}(s)/V_{in}(s)$

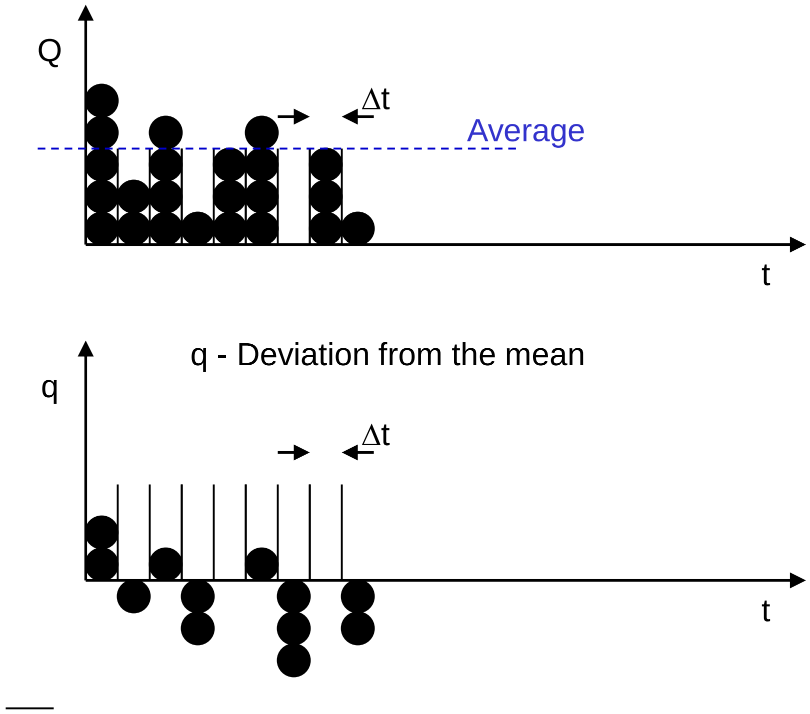
The circuit can also be represented using a pulse response. The representation with the help of pulse response is suitable if the circuit changes over time – e.g. it is switched on at a certain moment.

Thermal Noise in a Resistor

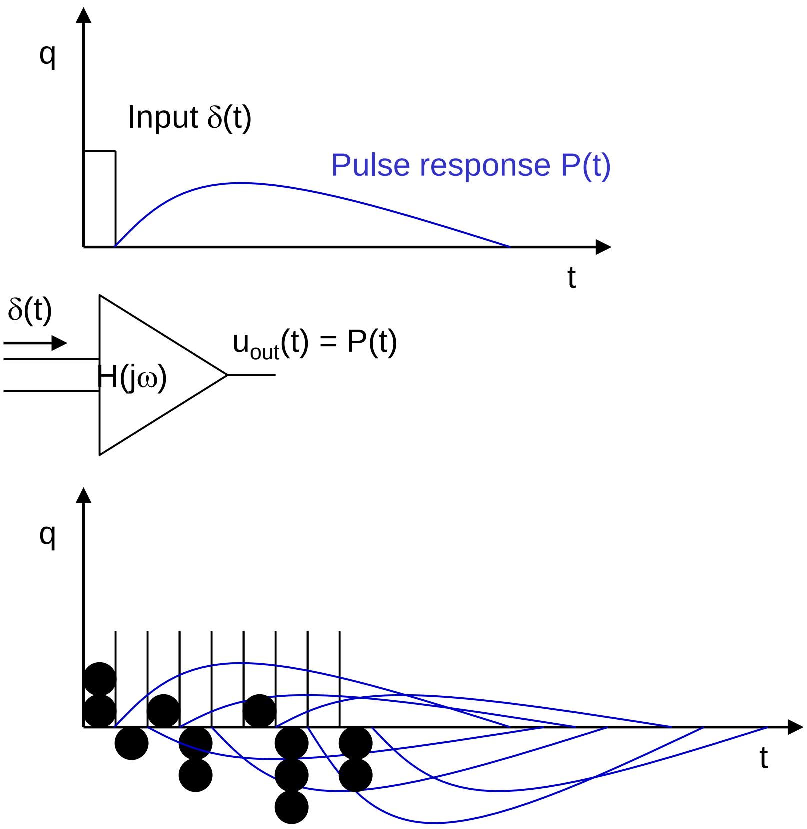
The electrons move in a resistor not only because of the field (drift) but also because they have kinetic (thermal) energy. This additional movement causes the fluctuations in the current - the noise. Let us try to calculate the noise.



We divide (quantize) time into short time intervals Δt. In each time interval, a certain number of electrons leave the resistor. The mean charge $\langle Q \rangle$ is equal to the current multiplied by Δt. Because of the noise, the actual charge fluctuates around the mean value. We refer to these fluctuations as q.



The (noise) charges q in short time intervals Δt can be understood as charge pulses. We assume that the resistor is connected to a circuit. This circuit has a pulse response and a transfer function. We want to calculate the noise signal at the output u_{out}. Signal u_{out} is generated by the noise signals q produced by the resistor.



Each input pulse generates an output signal. Each output signal is the product of the pulse response and the noise charge q(τ):

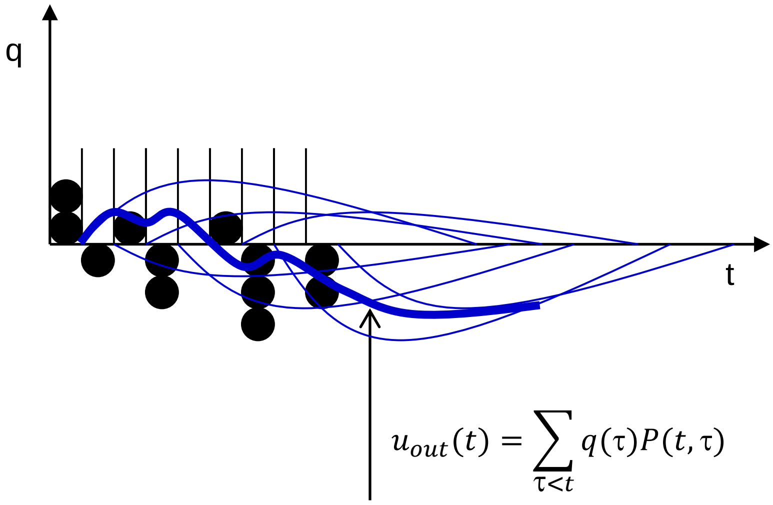
$u_{out,\tau}(t) = q(\tau)P(t,\tau)$

P(t, τ) is the pulse response at a moment t to an input pulse with integral 1 (Dirac pulse) that was generated at the moment τ.

The total amplitude of the output noise signal at a moment t is the sum of all output signals that were generated in the past:

$$u_{out}(t) = \sum_{\tau<t} q(\tau)P(t,\tau)$$

We don't know the actual values of noise charges q(τ) and can only calculate their variance. For this reason, it is only possible to calculate the variance of the noise signal at the output. This variance corresponds to the noise power.



The following formula can be derived for the noise in the time domain:

$$\langle u(t)^2\rangle = \int_{-\infty}^{t} \frac{\langle q^2\rangle}{\Delta t} (P(t,\tau))^2\, d\tau$$

Δt is the length of the time intervals.

P(t, τ) is the pulse response at moment t to an input pulse that was generated at the moment τ. The formula for noise in time domain is suitable for systems that change over time, e.g. the circuits that contain switches.

If our system does not change in time, we can also derive the corresponding formula in frequency domain:

$$\langle u^2\rangle = \int_0^{\infty} S(f)|H(i\omega)|^2\, df$$

S is the spectral power density of the noise source.

For white noise, it holds:

$$S(f) = 2\frac{\langle q^2\rangle}{\Delta t}$$

When S is independent of Δt and has no frequency dependence, we have white noise. In this case, the noise signal at the input can be modelled with arbitrarily short signals – Dirac pulses. Δt. We call the noise white because the Fourier transform from the input pulses (Dirac pulses) is a constant function. The noise at the input contains all frequencies, such as the white color, contains all wavelengths.

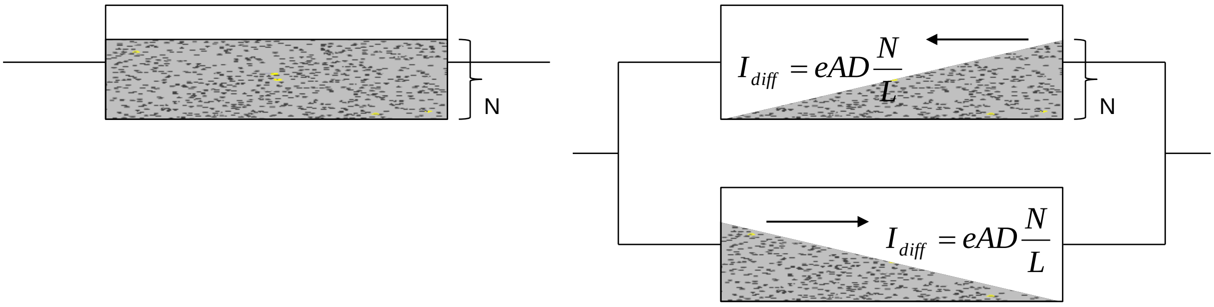
If the noise pulses at the input have a slower shape and S has a frequency dependence, the noise is not white. In this case, the following applies:

$$S(f) = 2\frac{\langle q^2\rangle}{\Delta t}|P_{in}(i\omega)|^2$$

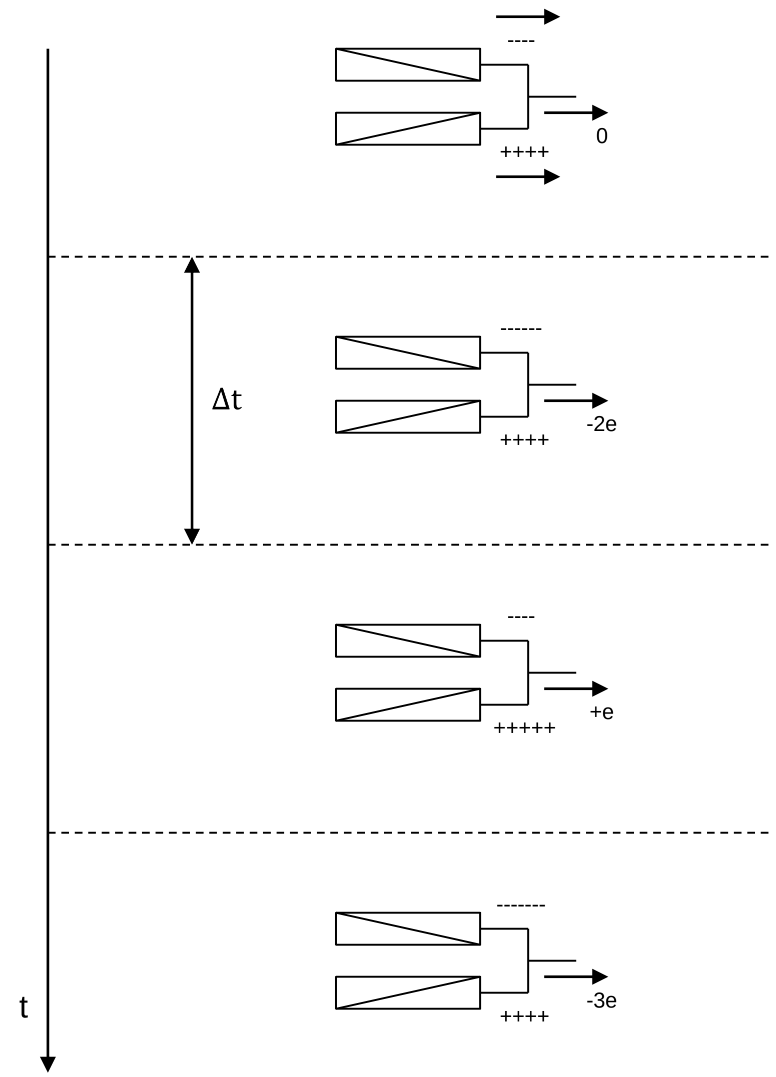
An example is the 1/f noise.

How large is the spectral power density for the thermal noise of a resistor? To answer this question, we need to calculate the variance of noise signals q. Here's how we do it.

We can split the resistance into two parallel resistors. They have uniformly decreasing charge densities. The total resistance of the parallel connection is R. A diffusion current flows in each partial resistor. The total diffusion current is zero because both subcurrents have the same magnitude and different directions.



The mean number of electrons that leaves each resistor-part in time is: $I_{diff}\Delta t/e$. We can assume that the actual number of electrons fluctuates around this mean.



The standard distribution for discrete fluctuations is Poisson distribution. The variance of this distribution is equal to the mean. For this reason, the variance of the charge q that leaves each resistor-part in Δt is:

$$< q^2 >= eI_{diff}\Delta t$$

The total variance is two times greater, because we have two resistor-parts.

If we consider the formula for diffusion current, the formula for resistance as a function of mobility, and the Einstein equation:

$$\frac{D}{\mu} = \frac{kT}{e}$$

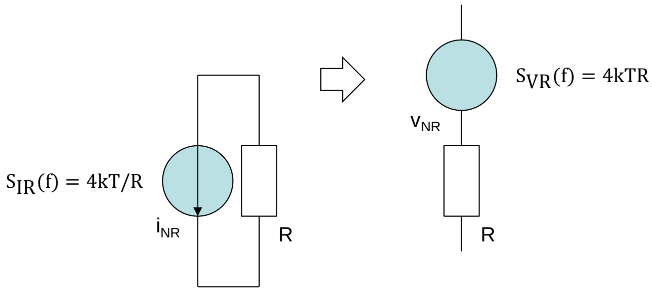
we get:

$$S_{IR}(f) = 4kT/R$$

Thermal noise is the result of diffusion currents (thermal movements) and discrete nature of the charge. (Poisson Statistics).

Poisson distribution is important for noise theory. Poisson distribution is the standard distribution for discrete random variables, just as the Gaussian distribution is the standard distribution for continuous random variables. Poisson distribution has the property that the variance (square of the standard deviation) is equal to the expected value (mean).

The noise of a resistor can be described with a current source with spectral power density S_{IR} = 4kT/R. The unit of S_{IR} is A^2s.



We can convert the current source with a parallel resistor into a voltage source with a series resistor. For the voltage and current of the respective sources, hold:

$U = I \times R$

The power density of the voltage source is $S_{VR} = R^2\, S_{IR}$.

We obtain:

$$S_{VR}(f) = 4kTR$$

The noise of a resistor can therefore also be described with a voltage source with a spectral power density of S_{VR} = 4kTR. The unit of S_{VR} is V^2s.

Noise in MOS transistor

The channel of the MOSFET can be seen of as a resistor with an uneven charge carrier density.

Thermal noise of the channel can be modelled with a current source. The power spectral density can be calculated using the equation for the resistor:

$$S_{IT} = \frac{4kT}{\langle R\rangle} \qquad (1)$$

1 / <R> is the average conductance of the channel:

$$\frac{1}{\langle R\rangle} = \frac{e\mu W\langle Q'\rangle}{L} \qquad (2)$$

<Q'> is the average charge/area (unit C/m^2). It can be calculated as follows:

$$\langle Q'\rangle = \frac{1}{L}\int_0^L Q'(x)dx \qquad (3)$$

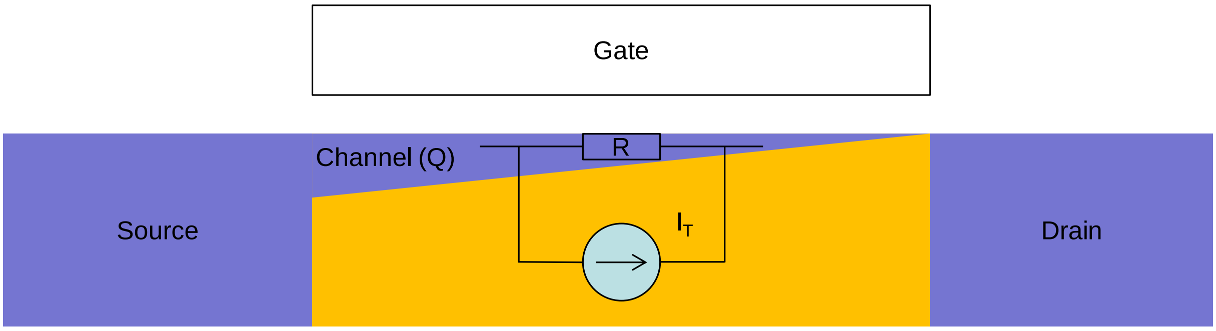


Figure 1: Thermal noise in the MOSFET channel

Substituting (3) into (2) and (1), we get the formula for the power spectral density:

$$S_{IT} = 4kT\mu\frac{W}{L}\langle Q'\rangle = \frac{4kTW\mu}{L^2}\int_0^L Q'(x)dx \qquad (4)$$

Let us assume that the transistor is in saturation.

It holds near source:

$$Q'(0) = C'_{ox}(V_{gs} - V_{th})$$

Near the drain, the charge is:

$$Q'(L) = 0$$

Exact calculation of the integral leads to:

$$\langle Q\rangle = \frac{2}{3}C'_{ox}\left(V_{gs} - V_{th}\right) \qquad (5)$$

(The exact derivation of equation (5) is in the next paragraph.)

Substituting (5) into (4), we get

$$S_{IT} = 4kT\mu\frac{W}{L}\frac{2}{3}C'_{ox}(V_{gs} - V_{th}) \qquad (6)$$

The transconductance of the transistor is given by the following formula:

$$g_m = \frac{dI_{dssat}}{dV_{gs}} = \frac{d}{dV_{gs}}\left(\frac{1}{2n}\mu C'_{ox}\frac{W}{L}(V_{gs}-V_{th})^2\right) = \frac{1}{n}\mu C'_{ox}\frac{W}{L}(V_{gs}-V_{th}) \qquad (7)$$

Substituting the formula for g_m (7) into that for S_{IT} (6), we obtain the final equation for the power spectral density of the current noise source in a MOSFET:

$$S_{IT} = 4kTn\frac{2}{3}g_m \qquad (8)$$

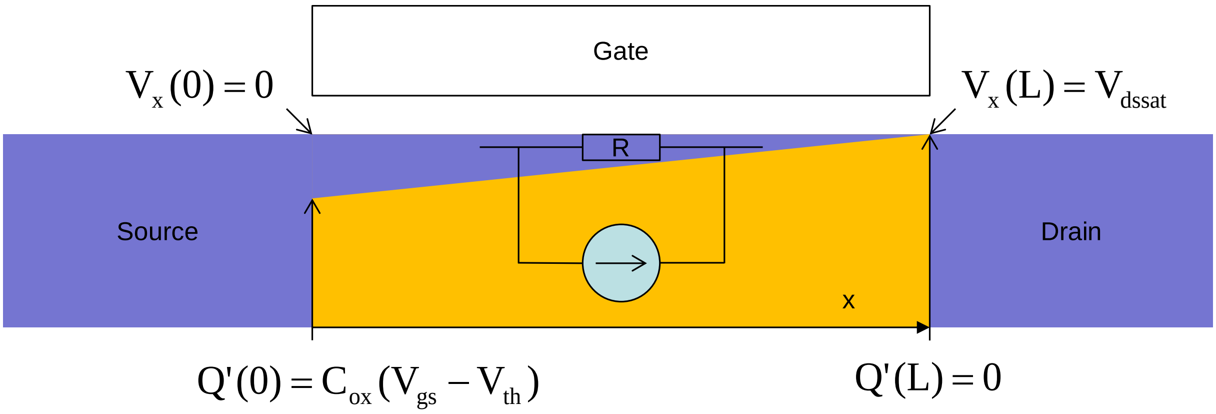


Figure 2: Thermal noise in the MOSFET channel

Derivation of the formula for transistor current I_{ds} and the average channel charge (optional)

In lecture 2 we had the following formula for the channel charge per area:

$$Q' = C'_{ox}(V_{gs} - V_{th}) \qquad (9)$$

The formula is valid in strong inversion for small V_{ds}.

The formula must be adjusted for larger V_{ds}. The channel charge per area varies from the value given by (9) on the source side to zero on the drain side, where the channel pinches off (Figure 2).

Let us denote the potential in the channel region (at the boundary between silicon and gate oxide) as $V_x(x)$. Coordinate x goes from source to drain (Figure 2). Let us define V_x with respect to the source potential V_s. That means:

$$V_x(0) = 0$$

and

$$V_x(L) = V_{ds}$$

The channel charge per area at any point x is:

$$Q'(x) = C_{ox}(V_{gs} - V_{th} - nV_x) \qquad (10)$$

Or simplified

$$Q'(x) = C_{ox}(V_{gst} - nV_x);\ V_{gst} \equiv V_{gs} - V_{th} \qquad (11)$$

This paragraph is a repeat from Lecture 3.

First, let us derive the equation for I_{ds} current for any V_{ds}.

We start with the formula for drift current:

$$I_{ds} = \mu\, W\, Q'(x)\, |E_x| = \mu\, C'_{ox}\, W\, (V_{gst} - nV_x)\, |E_x| \qquad (12)$$

E_x is the E-field component in the x-direction, W is the gate width, μ is the mobility of the charge carriers.

The following applies:

$$|E_x| = \frac{dV_x}{dx} \qquad (13)$$

Substituting (13) into (12), we get:

$$I_{ds} = \mu\, C'_{ox}\, W\, (V_{gst} - nV_x)\frac{dV_x}{dx}$$

or

$$I_{ds}dx = \mu\, C'_{ox}\, W\, (V_{gst} - nV_x)dv_x \qquad (14)$$

We can integrate the two sides:

$$\int_0^L I_{ds}dx = \int_0^{V_{ds}} \mu\, C'_{ox}\, W\, (V_{gst} - nV_x)dv_x$$

It follows:

$$I_{ds} = \frac{1}{2n}\mu\, C'_{ox}\frac{W}{L}\left[V_{gst}{}^2 - (V_{gst} - nV_{ds})^2\right]$$

After squaring the second term, we get the general formula for transistor current:

$$I_{ds} = \mu\, C'_{ox}\frac{W}{L}\left[V_{gst}V_{ds} - n\frac{V_{ds}^2}{2}\right] \qquad (15)$$

In saturation:

$$V_{ds} = V_{dssat} = \frac{V_{gst}}{n} \qquad (16)$$

Substituting this into (15), we get the formula for saturation current:

$$I_{dssat} = \frac{1}{2n}\mu\, C'_{ox}\frac{W}{L}\, V_{gst}^2 \qquad (17)$$

Now let us calculate the average channel charge in a transistor in saturation:

$$\langle Q' \rangle = \frac{1}{L}\int_0^L Q'(x)dx$$

Let us start with the integral:

$$\int_0^L Q'(x)dx = \int_0^L C'_{ox}(V_{gst} - nV_x)dx$$

The integral is the total charge in the channel divided by W.

Since we do not know the function $V_x(x)$, we change the integration variable from x to Vx (substitution):

$$\frac{Q}{W} = \int_0^L C'_{ox}\left(V_{gst} - nV_x\right)dx = \int_0^L C'_{ox}\left(V_{gst} - nV_x\right)\frac{dV_x}{dV_x/dx} = \int_0^{V_{dssat}} C'_{ox}\left(V_{gst} - nV_x\right)\frac{dV_x}{|E_x|}$$

The E field can be calculated using (12):

$$I_{ds} = \mu\, C'_{ox}\, W\left(V_{gst} - nV_x\right)|E_x|; \Rightarrow |E_x| = \frac{I_{ds}}{\mu C'_{ox}\, W\left(V_{gst}-nV_x\right)} \qquad (18)$$

It follows:

$$\frac{Q}{W} = \int_0^{V_{ds}} C'_{ox}\left(V_{gst} - nV_x\right)\frac{dV_x}{|E_x|} = \frac{\mu\left(C'_{ox}\right)^2 W}{I_{ds}}\int_0^{V_{dssat}}\left(V_{gst} - nV_x\right)^2 dV_x$$

$$= \frac{\mu\left(C'_{ox}\right)^2 W}{nI_{ds}}\left[\frac{\left(V_{gst}-nV_{dssat}\right)^3}{3} - \frac{\left(V_{gst}\right)^3}{3}\right] = \frac{\mu\left(C'_{ox}\right)^2 W}{nI_{ds}}\frac{\left(V_{gst}\right)^3}{3} \qquad (19)$$

In saturation, $I_{ds} = I_{dssat}$. Substituting (17) into (19), we get:

$$\frac{Q}{W} = \frac{\mu\left(C'_{ox}\right)^2 W}{nI_{ds}}\frac{\left(V_{gst}\right)^3}{3} = \frac{2}{3}LC'_{ox}V_{gst} \qquad (20)$$

Therefore it is:

$$\langle Q'\rangle = \frac{1}{L}\int_0^L Q'(x)dx = \frac{2}{3}C'_{ox}V_{gst} \qquad (21)$$

Example voltage amplifier

Let us consider the voltage amplifier as in lecture 6. We consider the variant with the MOSFET amplifier and resistor R_f. We use resistor R_f for DC feedback.

In Lecture 6 we derived the transfer function

$$V_o(s) = -V_s(s)\frac{C_i}{C_f}\alpha\frac{1}{(sT_r+1)}\frac{sT_f}{(sT_f+1)} \qquad (22)$$

The time constants are:

$$T_r = \frac{sC'_o\alpha}{\beta g_m} = \frac{s(C_iC_o+C_iC_f+C_fC_o)}{C_f g_m} \qquad (23)$$

with

$$C'_o = C_o + \frac{C_i^+C_f}{C_i^++C_f},\ \beta = \frac{C_f}{C_i^++C_f}, \alpha \equiv \frac{\beta\times g_m R_0}{1+\beta\times g_m R_0}, C_i^+ = C_i + C_g$$

And

$$T_f = R_f C_f \qquad (24)$$

If we have a voltage step with amplitude 1 at the input, the output voltage magnitude increases within $3 \times T_r$ to about C_i/C_f and decreases again to 0 (Figure 3 above). (Gain is negative.)

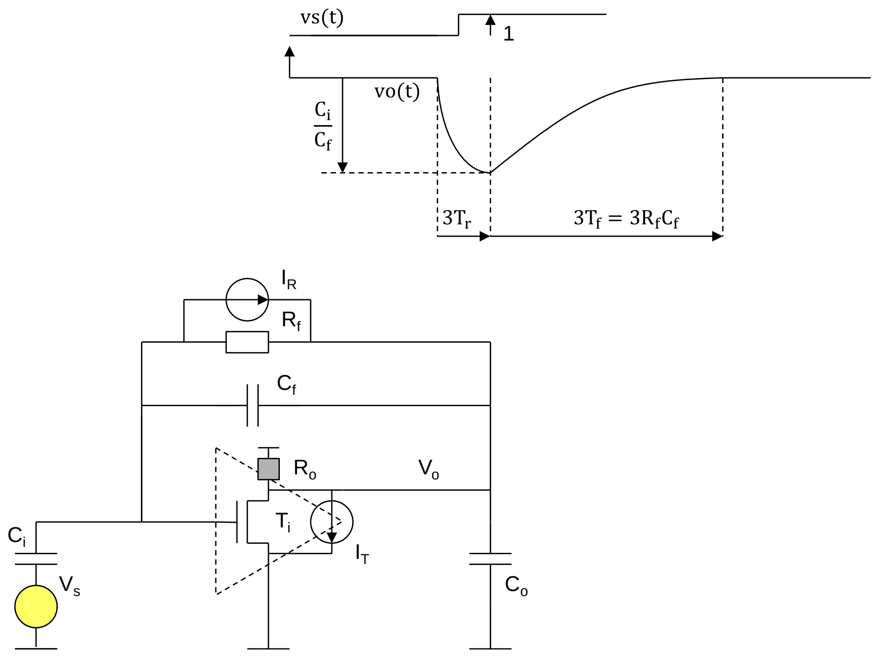


Figure 3: Voltage amplifier with noise sources. The figure above shows the step response.

The most important noise sources are the transistor T_i (noise source I_T) and the resistor R_f (noise source I_R). The components connected to the input of the amplifier generate large noise at the output, because the noise is amplified. Resistor R_o contributes little to the noise since it is connected directly to the amplifier output, and its noise is not amplified. Figure 3 shows the circuit with the noise sources I_T and I_R (white) and the input signal source V_s (yellow), which is switched off in further analysis. We will perform the frequency dependent noise analysis.

Step 1

We can move the transistor noise source to the input of the amplifier (Figure 4).

The following idea helps us: noise of the transistor can be modelled either with the current source IT between the drain and the source or with the voltage source V_T at the gate. Since an additional voltage at the gate - v_g - produces a drain-source current $g_m \times v_g$, and since the power densities are proportional to the square of the voltage or current, the following holds:

$$S_{VT} = \frac{S_{IT}}{g_m^2} = \frac{4\frac{2}{3}kTn}{g_m} \qquad (25)$$

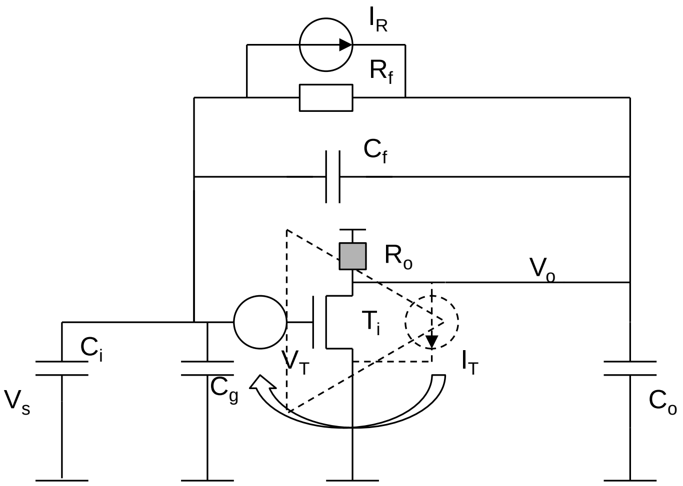
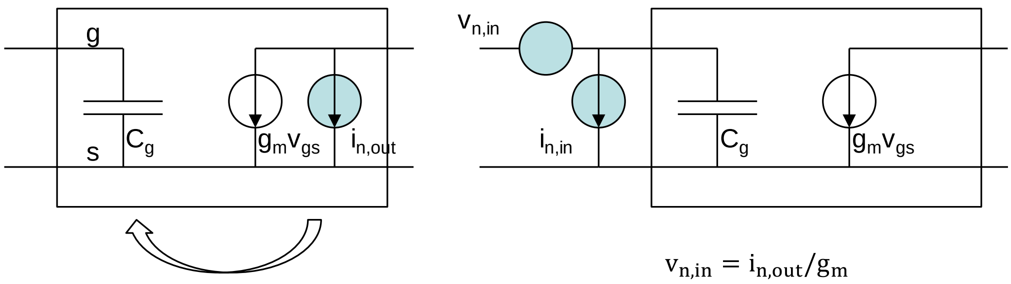


Figure 4: The noise source I_T can be moved to transistor input

Let us note the following:

If we move the noise sources to the input of a two-port circuit, and if that circuit has a finite input impedance, we must use a voltage source in series and a current source in parallel with the input.



$$v_{n,in} = i_{n,out}/g_m$$

$$i_{n,in} = \frac{v_{n,in}}{Z_{in}} = sC_g v_{n,in}$$

Figure 5: Moving of the noise source - C_g inside two-port

However, we will define the gate and source, as the input of the transistor for noise analysis, after the C_{gs}. The two-port then has an infinite input impedance and the parallel current noise source has zero amplitude. A voltage source is sufficient for modelling the noise.

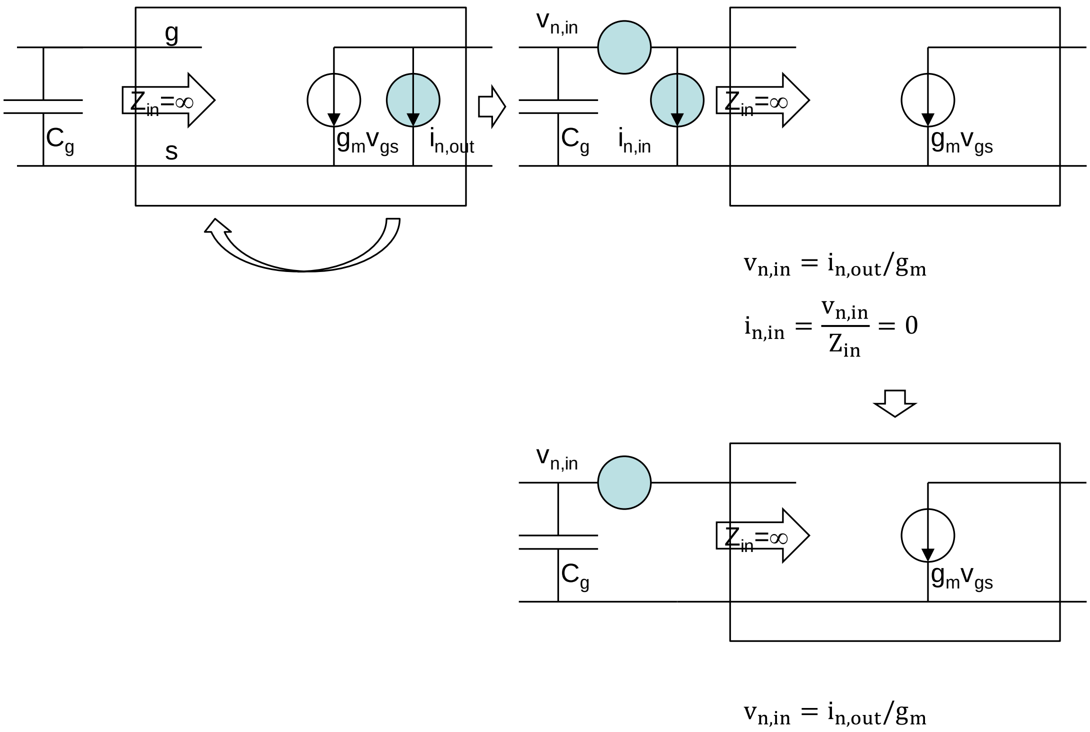


Figure 6: Moving of noise source - C_g in front of the two-port

Step 2: Moving of the Source I_R

The current source I_R is connected between the input and the output of the amplifier (Figure 3). The current that the noise source directs into the output produces little noise. The noise current that flows into the input is amplified by the amplifier and generates a large noise at the amplifier output. It is therefore equal for the noise calculation whether the source is connected between the input and the output of the amplifier or between the input of the amplifier and ground. We will connect the source I_R between the amplifier-input and the ground as this makes it easier to calculate the transfer function.

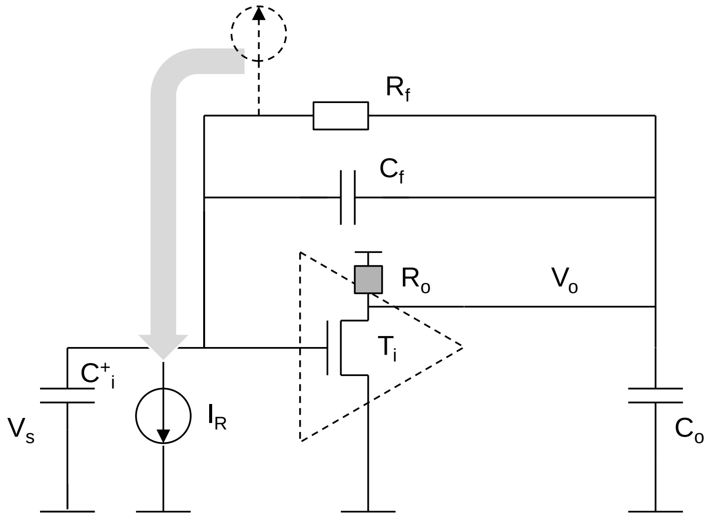


Figure 7: Moving of the noise source I_R

Step 3

To calculate the variance (power) of the noise signal at the amplifier-output, we will use the formula for noise in the frequency domain. If there are multiple sources of noise, the contributions can be counted separately. The total noise power is the sum of individual contributions:

$$< v_o^2 > = \int_0^{\infty} (S_{IR}(f)|H_{IR}(i\omega)|^2 + S_{VT}(f)|H_{VT}(i\omega)|^2)df \quad (26)$$

We now need to derive the transfer functions H(iω) for two noise sources.

Step 4

Transfer function for the I_R source.

We will perform a simplified analysis and assume that we have virtual ground at point v_i (Figure 8).

$$H(i\omega) = \frac{dV_o}{dI_R} = -Z_f = -\frac{R_f}{1+sR_fC_f} \quad (27)$$

Contribution of the source I_R to the total noise is:

$$\langle v_{oIR}\rangle^2 = \int_0^{\infty} S_{IR}\left|\frac{R_f}{1+i\omega T_f}\right|^2 df = S_{IR}\frac{R_f^2}{2\pi T_f}\int_0^{\infty}\left|\frac{R_f}{1+i\omega T_f}\right|^2 d(\omega T_f) = S_{IR}\frac{R_f^2}{2\pi T_f}\tan^{-1}(\omega T_f)]_0^{\infty} = S_{IR}\frac{R_f^2}{2\pi T_f}\frac{\pi}{2} = S_{IR}\frac{1}{4}\frac{R_f}{C_f} \quad (28)$$

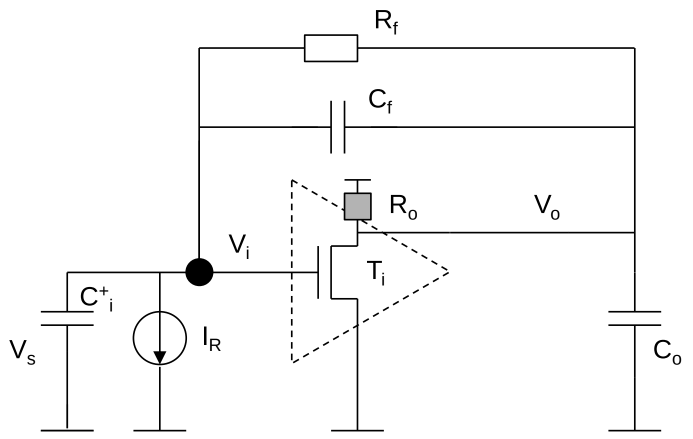


Figure 8: Noise source I_R

By substituting the power spectral density formula, we get:

$$< v_{oIR} >^2 = \frac{1}{4}S_{IR}\frac{R_f}{C_f} = \frac{1}{4}\frac{R_f}{C_f}\frac{4kT}{R_f} = \frac{kT}{C_f} \quad (29)$$

Figure 9 shows the Bode diagram of the transfer function $|H|^2$. The yellow area contributes significantly to the integral in (28).

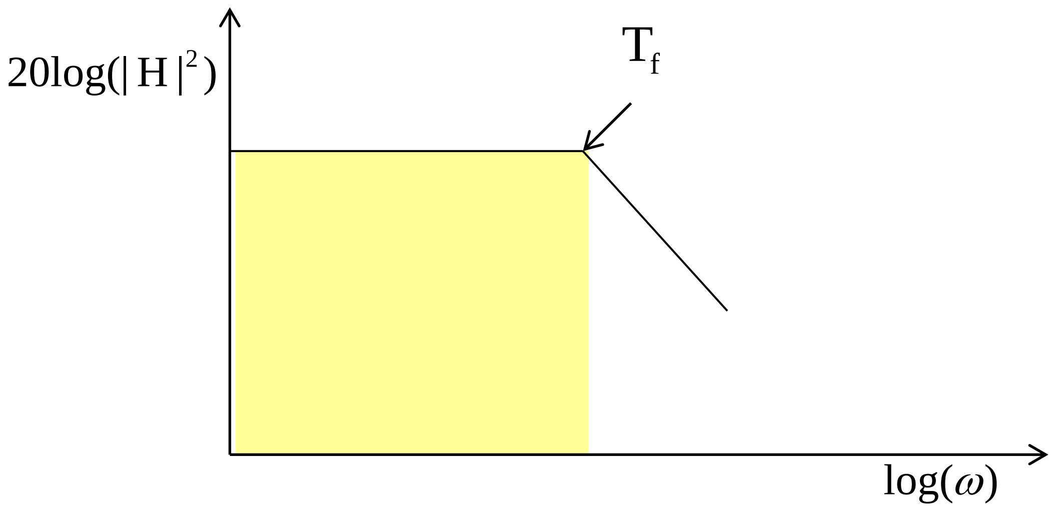


Figure 9: Bode plot of the H_{IR} transfer function

The formula for the noise power caused by the source I_R can be rewritten as follows:

$$< v_{oIR} >^2 = \frac{1}{4}\frac{1}{C_f^2} S_{IR} T_f \qquad (30)$$

We use the result $T_f = R_f\,C_f$.

This formula is easy to remember: the noise power depends on the power spectral density S multiplied by the time constant T_f.

Step 5

For the voltage source V_T we can derive the transfer function as follows: We assume that v_g is virtual ground (Figure 10). It follows:

$$V_o = -\frac{Z_i + Z_f}{Z_i} V_T = -\frac{\frac{R_f}{1+i\omega R_f C_f} + \frac{1}{i\omega C_i}}{\frac{1}{i\omega C_i}} V_T = \frac{1+i\omega R_f (C_f + C_i^+)}{1+i\omega R_f C_f} V_T$$

$$C_i^+ = C_i + C_{gs}$$

The transfer function is:

$$H(i\omega) = \frac{V_o}{V_T} = \frac{1+i\omega R_f (C_f + C_i^+)}{1+i\omega R_f C_f} = \frac{1+i\omega R_f (C_f + C_i^+)}{1+i\omega T_f} \qquad (31)$$

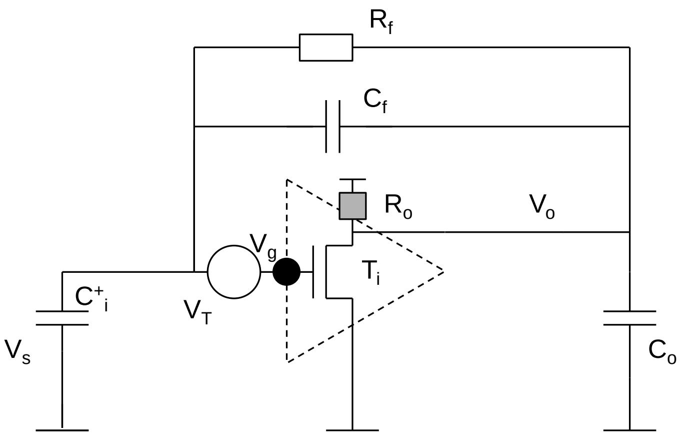


Figure 10: Noise source V_T

Note that this transfer function (31) predicts constant gain for high frequencies (Figure 11). If we substituted this function in formula (26), we would obtain infinite noise power. Something is wrong.

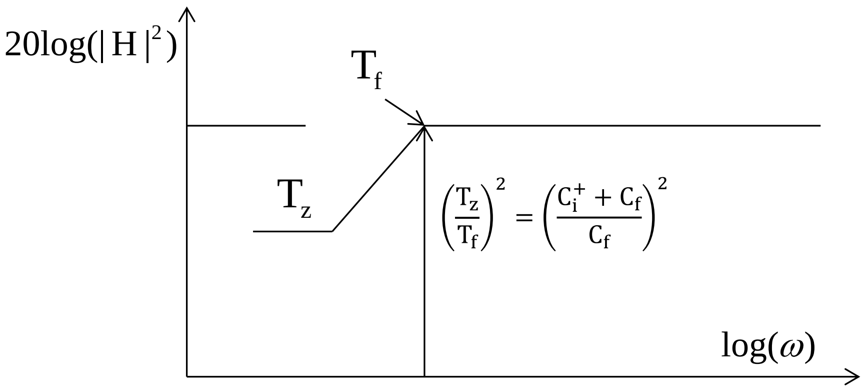


Figure 11: Bode plot of the H_{IR} transfer function, case without Tr

For high frequencies, the virtual ground assumption no longer applies. We did a more detailed analysis of the circuit in Lecture 6 and we derived two time constants in the transfer function:

$$T_r = \frac{sC'_o\alpha}{\beta g_m} \qquad (32)$$

$$T_f = R_f C_f \qquad (33)$$

The circuit in Lecture 6 is the same as the circuit in Figure 10 except for the position of the input source. The position of the input source does not change the time constants in the denominator of the transfer function (the poles).

We can extend formula (31) by adding the second time constant:

$$H(i\omega) = \frac{V_o}{V_T} = \frac{(1+i\omega T_Z)}{(1+i\omega T_r)(1+i\omega T_f)} \qquad (34)$$

with:

$$T_r = \frac{sC'_o\alpha}{\beta g_m}$$

$T_f = R_f C_f$

$T_z = R_f(C_f + C_i^+)$

The following applies to the time constants:

$T_z > T_f > T_r$

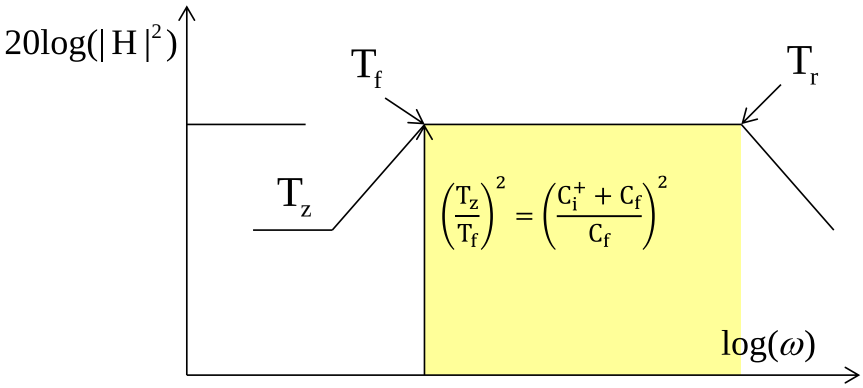


Figure 12: Bode plot of transfer function H_{IR}, case with T_r

The frequency range between $1/T_f$ and $1/T_r$ (yellow area in Figure 12) has the largest contribution to the integral. In this frequency region we can simplify the transfer function (34) as follows:

$$H(i\omega) \sim \frac{T_z}{T_f(1+i\omega T_r)} = \frac{C_f + C_i^+}{C_f} \frac{1}{1+i\omega T_r} \qquad (35)$$

It is then relatively easy to calculate the noise power:

$$< v_{oT} >^2 = \int_0^\infty S_{VT} \left(\frac{C_f + C_i^+}{C_f}\right)^2 \left|\frac{1}{1+i\omega T_r}\right|^2 df = S_{VT} \left(\frac{C_f + C_i^+}{C_f}\right)^2 \frac{1}{2\pi T_r} \frac{\pi}{2} = S_{VT} \left(\frac{C_f + C_i^+}{C_f}\right)^2 \frac{1}{4T_r}$$

$$< v_{oT} >^2 = \frac{4kTn2/3}{g_m}\left(\frac{C_f+C_i^+}{C_f}\right)^2 \frac{1}{4T_r} = \frac{kTn2/3}{g_m}\left(\frac{C_f+C_i^+}{C_f}\right)^2 \frac{1}{T_r} \qquad (36)$$

The formula for the noise power due to source V_T can be rewritten as:

$$< v_{oT} >^2 = \frac{1}{4}\left(\frac{C_f+C_i^+}{C_f}\right)^2 \frac{S_{VT}}{T_r} \qquad (37)$$

Summary

The variance of the noise signal at the output of the voltage amplifier is given by the following formulas:

$$< v_o >^2 = < v_{oR} >^2 + < v_{oT} >^2 = \frac{1}{4}\frac{1}{C_f^2}\left((C_i^+ + C_f)^2 \frac{S_{VT}}{T_r} + S_{IR}T_f\right) \qquad (38)$$

$$S_{IR} = \frac{4kT}{R_f} \qquad (39)$$

$$S_{VT} = \frac{4kTn2/3}{g_m} \qquad (40)$$

Let us now calculate the Signal to Noise Ratio (SNR).

A voltage step at the input with amplitude V_{isig} causes an output signal with amplitude: (Figure 13)

$$V_{osig}^2 = V_{isig}^2 \left(\alpha \frac{C_i}{C_f}\right)^2 \sim V_{isig}^2 \left(\frac{C_i}{C_f}\right)^2 \quad (41)$$

Signal to noise ratio is:

$$\frac{<v_o>^2}{V_{osig}^2} = \frac{1}{SNR^2} = \frac{1}{V_{osig}^2} \frac{1}{4} \frac{1}{C_f^2} \left((C_i^+ + C_f)^2 \frac{S_{VT}}{T_r} + S_{IR} T_f\right) \quad (42)$$

or

$$\frac{<v_o>^2}{V_{osig}^2} = \frac{1}{SNR^2} = \frac{1}{V_{isig}^2} \frac{1}{4} \frac{1}{C_i^2} \left((C_i^+ + C_f)^2 \frac{S_{VT}}{T_r} + S_{IR} T_f\right) \quad (43)$$

An "equivalent noise signal" (equivalent noise voltage - ENV) can be defined as a signal at the input that leads to SNR = 1.

$$ENV^2 = \frac{1}{4} \frac{1}{C_i^2} \left((C_i^+ + C_f)^2 \frac{S_{VT}}{T_r} + S_{IR} T_f\right)$$

This ENV is equal to "input referred noise".

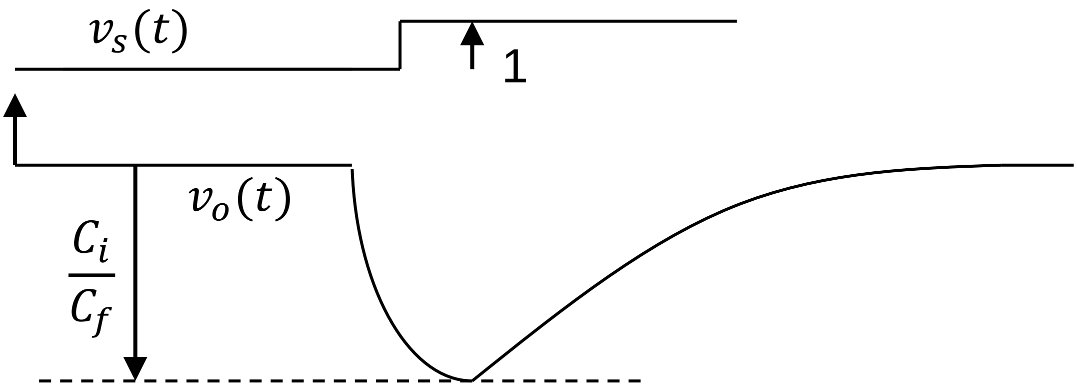


Figure 13: Step response

Discussion

We would like to optimize the amplifier's parameters to achieve maximum SNR.

What sizes can we vary? We want the gain $A = C_i/C_f$ to remain constant during optimization.

We can increase C_i and C_f by the same factor. C_i and C_f magnification helps to reduce R_f noise contribution (30):

$$ENV_R^2 = \frac{1}{4} S_{IR} \frac{T_f}{C_i^2} = \frac{1}{4A^2} \frac{R_f}{C_f} \frac{4kT}{R_f} = \frac{kT}{A^2 C_f} \quad (44)$$

Transistor noise contribution remains unchanged (37):

$$ENV_T^2 = \frac{kT2/3}{g_m} \left(\frac{C_f + C_i^+}{C_i}\right)^2 \frac{1}{T_r} \quad (45)$$

We can enlarge C_o. Larger C_o makes the amplifier slower and helps to reduce the contribution of the transistor noise. This follows from the formula for the time constant:

$$T_r = \frac{sC'_o \alpha}{\beta g_m} \quad (46)$$

We can increase the transconductance g_m by multiplying the transistor and R_o. Larger g_m (for constant C_o) makes the transistor faster and leaves the noise unchanged. (See the formula 37 and the formula for the time constant T_r (45).

Optimizing the amplifier to achieve required SNR, gain and bandwidth (summary)

Let us start with the "minimum size" C_f (e.g. 10 fF) and with a small transistor and small transistor current (e.g. 10 µA).

First, we scale up C_f and C_i until the R_f - noise contribution becomes small enough.

We increase C_o until the transistor noise contribution becomes small enough.

After this, we check whether the amplifier has the required bandwidth i.e. is fast enough.

If the amplifier is too slow, we increase g_m by multiplying the transistor and the R_o.

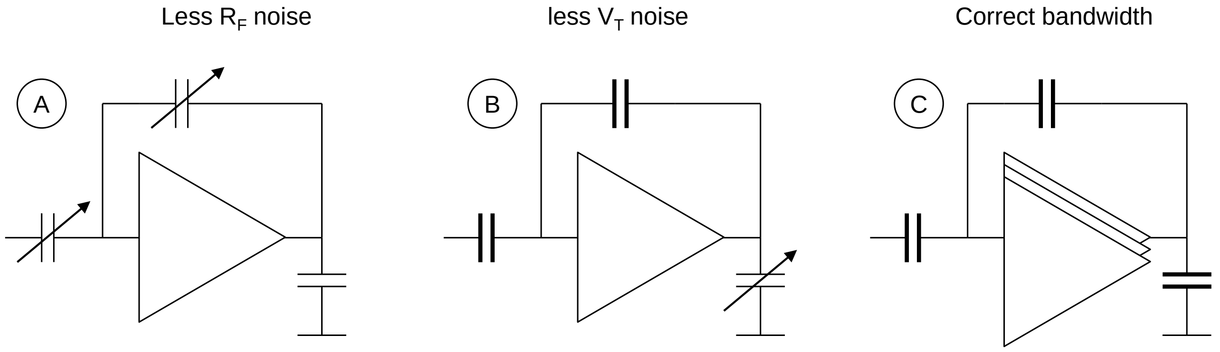


Figure 14: Optimization method

This method can result in too large $C_i^+ = C_g + C_i$ and too large power consumption.

The following applies to most analogue circuits (see e.g. formula 37):

$SNR^2 \sim 1/T_r$

$SNR^2 \sim 1/g_m$ = 1/ power consumption

Sometimes, the following figure of merit is defined as a measure of how well a circuit has been optimized:

FOM = power consumption Tr $/SNR^2$

This figure of merit should be similar for all well optimized circuits.

Circuit with smaller FOM is better. The figure of merit allows us to compare different circuits.

Charge sensitive amplifier

Charge sensitive amplifier is an important circuit. The input signal is a current pulse with charge Q, the output signal is proportional to Q and independent of the shape of the pulse.

A charge sensitive amplifier (Figure 15) has the same form as a voltage amplifier.

Let us derive the transfer function of the charge sensitive amplifier.

We start with the transfer function of the voltage amplifier.

The transfer function of the voltage amplifier is:

$$V_o(s) = -V_s(s)\frac{C_i}{C_f}\alpha\frac{1}{(sT_r+1)}\frac{sT_f}{(sT_f+1)} \qquad (47)$$

The input voltage source (Figure 3) can be converted to an equivalent current source (Figure 15). The following applies to the current of the current source and the voltage of the voltage source:

$$\frac{i_s(s)}{sC_i} = v_s(s) \qquad (45b)$$

In the time domain, the current i(t) is the integral of the voltage v(t) divided by C_i.

If we substitute (45b) into (22), we get the transfer function of the charge amplifier:

$$V_o(s) = -\frac{i_s(s)}{sC_f}\alpha\frac{sT_f}{(sT_r+1)(sT_f+1)} = -\frac{Q}{C_f}\alpha\frac{1}{(sT_r+1)}\frac{sT_f}{(sT_f+1)} \qquad (45c)$$

From this it follows: A current pulse with the charge Q generates an output signal with the amplitude:

$$V_{osig} \sim \frac{Q_{isig}}{C_f} \qquad (48)$$

(Figure 15) The output voltage increases to the maximum amplitude (Q/C_f) within $3 \times T_r$ and goes back down to 0. (The gain is negative.)

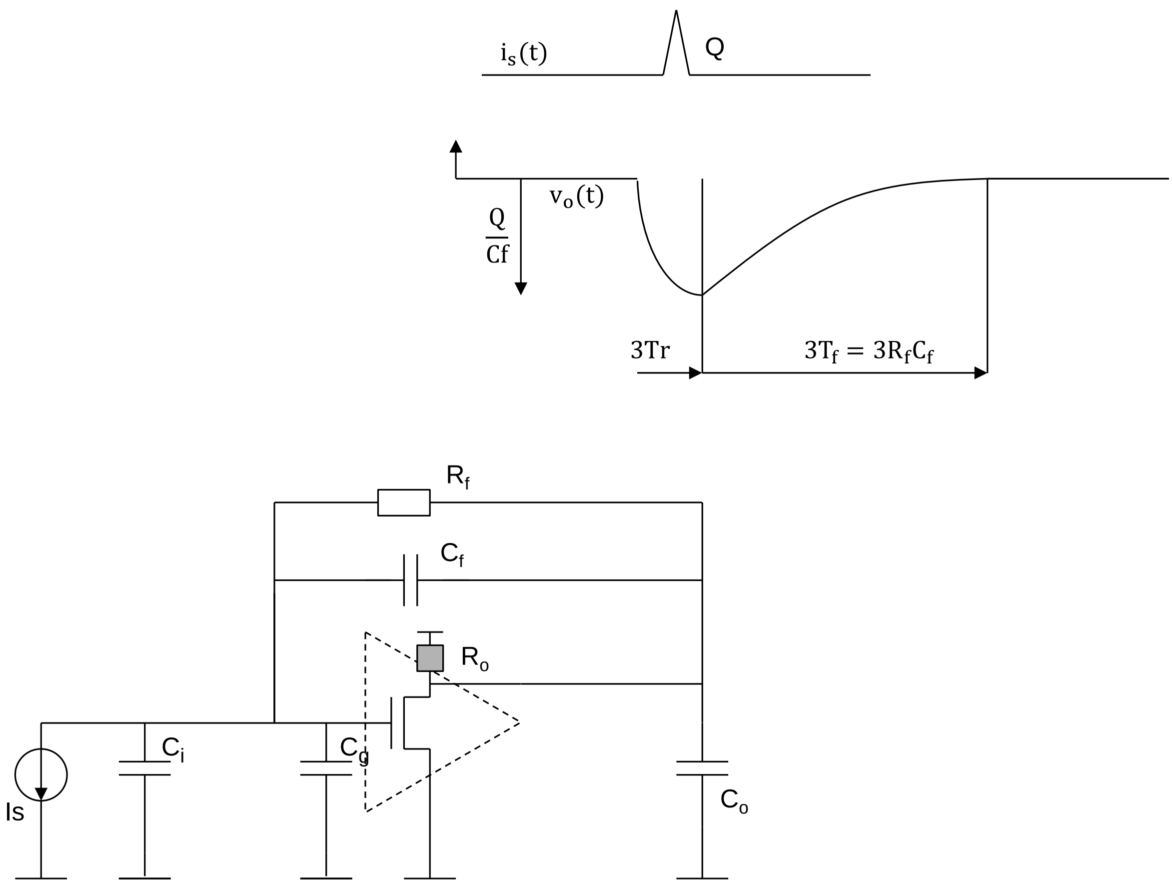


Figure 15: Charge sensitive amplifier. Top - pulse response

There are three main sources of noise in the circuit (Figure 16): the input transistor (source V_T), the resistor R_f (source I_R).

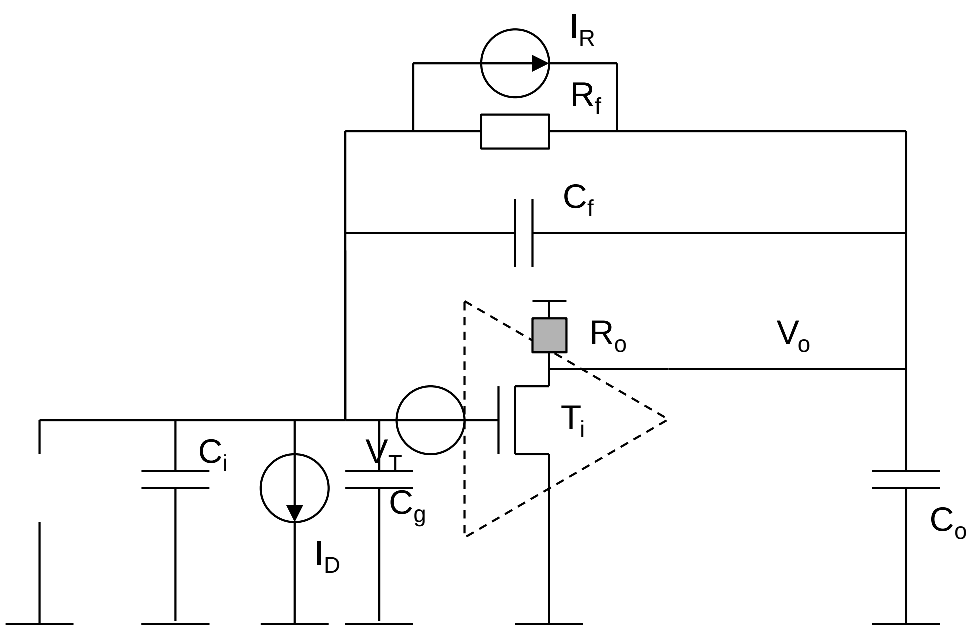


Figure 16: Charge sensitive amplifier. Sources of noise

The third source we have in the case that a sensor, having a leakage current, is connected to the amplifier as a signal source (noise source I_D).

Since the circuit has the same form as the voltage amplifier, the same formulas also apply.

Therefore, the variance of the noise signal at the output is:

$$< v_o >^2 = \frac{1}{4}\frac{1}{C_f^2}\left((C_i^+ + C_f)^2 \frac{S_{VT}}{T_r} + S_{IR}T_f + S_{ID}T_f\right) \qquad (49)$$

The power spectral densities are

$$S_{VT} = \frac{4kTn2/3}{g_m}$$

(thermal noise)

$$S_{IR} = \frac{4kT}{R_f}$$

(thermal noise)

$$S_{ID} = 2eI_{leak}$$

(leakage current noise)

Signal to noise ratio (SNR) is:

$$\frac{<v_o>^2}{V_{osig}^2} = \frac{1}{SNR^2} = \frac{1}{V_{osig}^2}\frac{1}{4}\frac{1}{C_f^2}\left((C_i^+ + C_f)^2 \frac{S_{VT}}{T_r} + S_{IR}T_f\right) \qquad (50)$$

with

$$V_{osig} = \frac{Q_{isig}}{C_f}$$

Therefore:

$$\frac{<v_o>^2}{V_{osig}^2} = \frac{1}{SNR^2} = \frac{1}{Q_{isig}^2}\frac{1}{4}\left((C_i^+ + C_f)^2 \frac{S_{VT}}{T_r} + S_{IR}T_f\right) \qquad (51)$$

Equivalent noise signal (charge), defined as the charge signal at the input that leads to SNR = 1:

$$ENC^2 = \frac{1}{4}\left((C_i^+ + C_f)^2 \frac{S_{VT}}{T_r} + S_{IR}T_f + S_{ID}T_f\right) \qquad (52)$$

An input signal = ENC produces at the output a signal with the amplitude equal to the standard deviation of the noise signal. It is roughly the smallest measurable signal if the measurement is repeated just once.

Optimization of the charge amplifier

Let us optimize the parameters of the charge amplifier to achieve maximum SNR.

It is advantageous to minimize the input capacitance (the sensor capacitance). This reduces the noise contribution of the transistor.

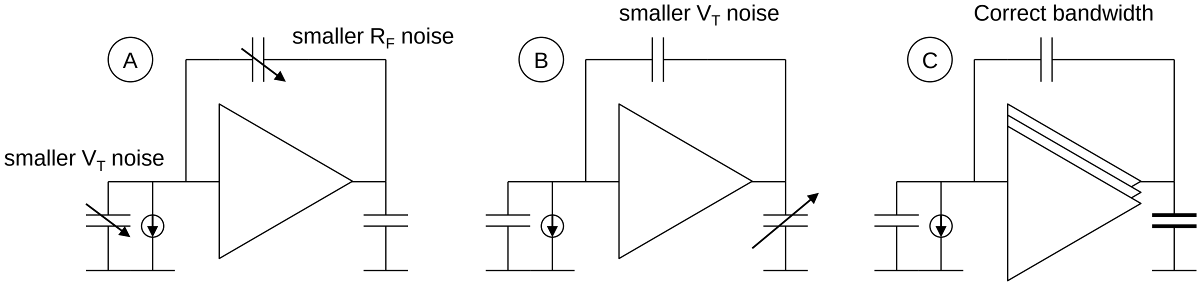
We can also choose a small C_F to reduce the R_F noise contribution:

$$ENC_R^2 = \frac{1}{4}S_{IR}R_fC_f$$

We can also enlarge C_o. Larger C_o makes the amplifier slower (T_r increases) and helps with transistor noise:

$$T_r = \frac{sC'_o}{\alpha\beta g_m}$$

We can increase the transconductance g_m by multiplying the transistor and R_o. Larger g_m (at constant C_o) makes the transistor faster and leaves the noise unchanged.



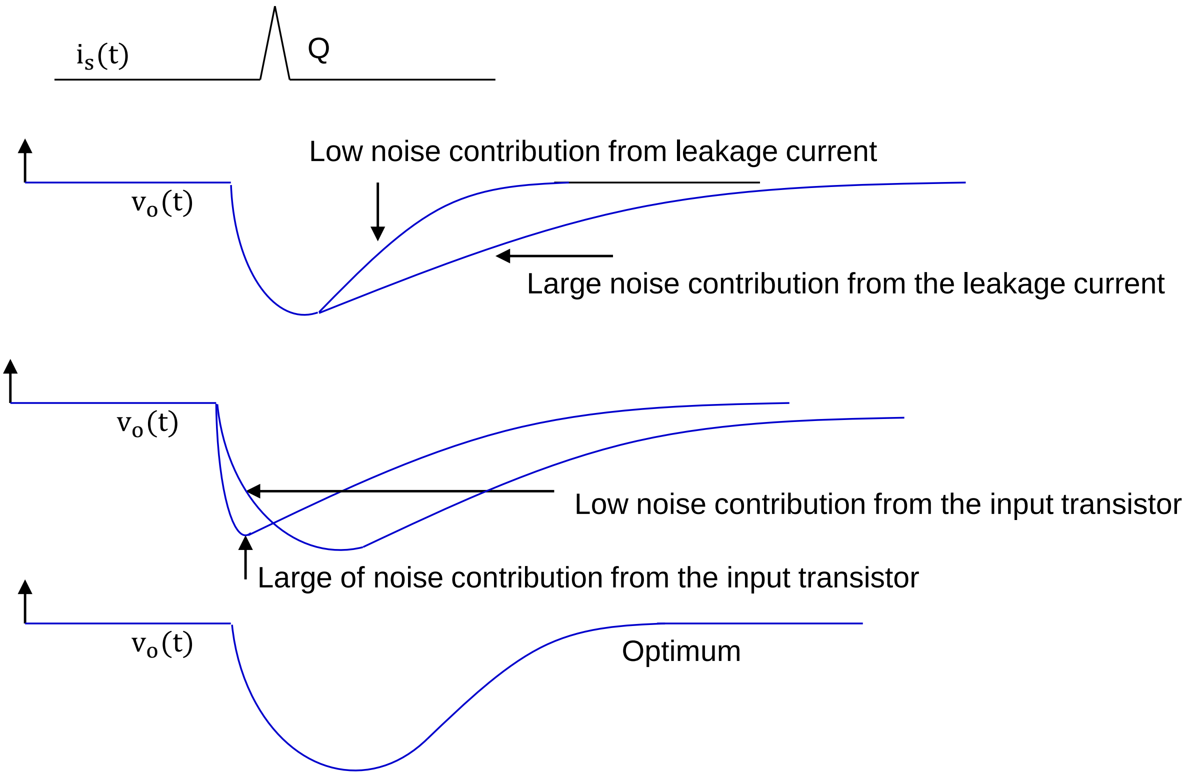


Figure 17: Noise and time constants

Figure 17 illustrates the optimization of the charge-sensitive amplifier for maximum SNR.

It is important to minimize C_i.

Short time constant T_f minimizes the sensor leakage current noise contribution.

Slow time constant T_r is good to reduce the transistor noise contribution.

It can be shown that SNR is maximal when both time constants are similar in size.

1/f noise and transistor mismatch

We will consider two effects that at first glance appear to be quite different.

If we make measurements on a group of transistors with identical layouts, we would find that the drain currents are slightly different even if all other parameters such as voltages and temperature are the same. The currents are not equal, we have a "mismatch". The mismatch is time-independent, it is a permanent mismatch. The mismatch is described by its variance. The actual current value is a random variable, described by its mean value and variance.

1/f noise (or flicker noise) is the variation in transistor current over time.

Both effects 1/f noise and mismatch can be described as fluctuations of the charge in the transistor channel.

These fluctuations are caused by so-called trap states in the gate oxide near the silicon.

Let us consider such a trap. The trap can be described by two properties.

First, we have a probability $c_p\Delta t$ per time interval Δt that a charge carrier will be caught in the trap. C_r stands for capture rate. Second, we have the emission time τ. In average, the charge carrier spends the time τ in the trap.

Now let us consider a larger number of traps N_t. The mean number of trapped charge carriers in time Δt is:

$\langle \Delta N \rangle = N_t c_r \Delta t$

These charge carriers are released with a time constant τ. After time t, the number of trapped charge carries is:

$\Delta N(t) = \Delta N e^{-\frac{t}{\tau}}$

We obtain the total number of captured charge carriers as the integral:

$$< N_{trapped} >= \int_{-\infty}^{t} N_t c_r \, e^{-\frac{t-u}{\tau}} du = N_t c_r \tau$$

Transistor current in strong inversion depends on the total charge in the channel. The MOS structure is electro-neutral, therefore the sum of charges below the gate electrode is equal to the charge in the gate electrode. Trapping of electrons reduces the amount of charge in the channel. The transistor current decreases because the conductivity of the channel decreases. As the electrons get released, the current recovers. Since this process - catching and releasing electrons - is repeated over and over again, current noise is generated.

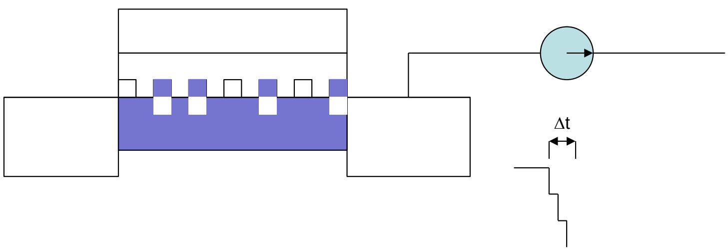


Figure 18: Charge carriers are trapped - current drops

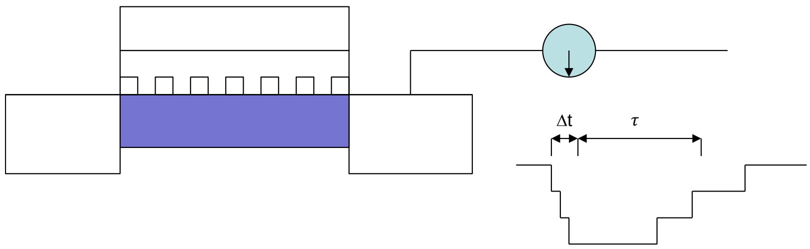


Figure 19: Charge carriers are released – current increases

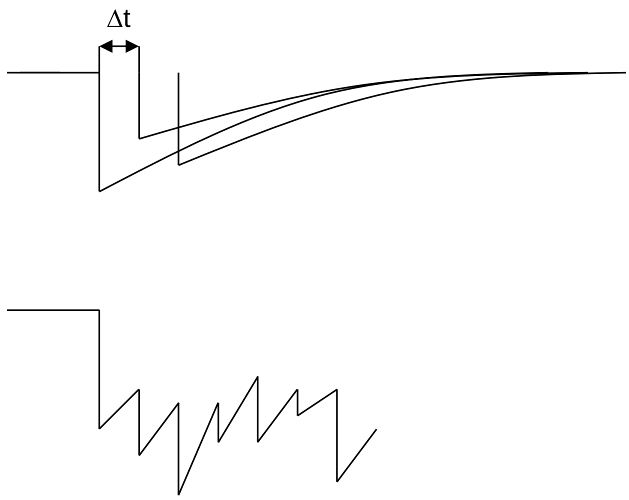


Figure 20: Trapping and releasing electrons creates noise

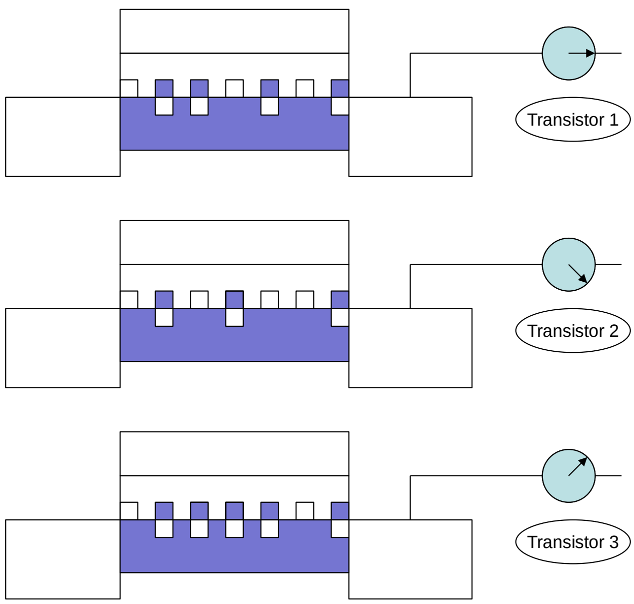


Figure 21: Mismatch. The number of trapped charge carriers varies from transistor to transistor

If we repeat measurements of the current through a transistor at different times, the respective current value depends on the number of currently trapped charge carriers. We assume that the measurements are far away in time and are thus independent.

To explain mismatch, we could assume that the charge carriers stay in the traps permanently. If we then repeat the current measurement on a transistor, we always get the same value. If we measure a group of several transistors, the current depends on the number of trapped charge carriers in the measured transistor. This number varies from transistor to transistor. Since in both cases the current fluctuation depends on the number of trapped charge carriers, with the only difference being that the noise is a fluctuation in time and the mismatch is a fluctuation within a group of transistors, we expect the same formula for the variance of the current.

Variance of the current

Let us calculate the variance of the transistor current.

The transistor current (strong inversion, saturation) is given by the following formula:

$$\mathrm{I_{dssat}} = \frac{1}{2\mathrm{n}}\,\mu\,\mathrm{C'_{ox}}\,\frac{\mathrm{W}}{\mathrm{L}}\,\mathrm{V_{gst}^2} \qquad (53)$$

Let us calculate the current as a function of the total charge in the channel. The charge is given by the following equation:

$$\mathrm{Q} = \frac{2}{3}\mathrm{LWC'_{ox}V_{gst}} \qquad (54)$$

Substituting this equation into (50), we obtain:

$$\mathrm{I_{dssat}} = \frac{3}{4\mathrm{n}}\frac{\mu}{\mathrm{L^2}}\,\mathrm{V_{gst}Q} = \mathrm{kQ} \qquad (55)$$

with

$$\mathrm{k} = \frac{3}{4\mathrm{n}}\frac{\mu}{\mathrm{L^2}}\,\mathrm{V_{gst}}$$

The change of channel charge dQ leads to the following change in current:

$$\mathrm{dI_{dssat}} = \mathrm{kdQ} \qquad (56)$$

The variance of this change is:

$$\langle \mathrm{dI_{dssat}^2} \rangle = k^2 \langle \mathrm{dQ^2} \rangle \quad (57)$$

The change in channel charge is caused by trapping of charge carriers.

$$\mathrm{dQ} = \mathrm{eN_{trapped}}$$

If we assume Poisson distribution, then if holds:

$$\langle \mathrm{dQ^2} \rangle = \mathrm{e^2}\,\langle \mathrm{N_{trapped}^2} \rangle = \mathrm{e^2} \langle \mathrm{N_{trapped}} \rangle = \mathrm{e^2 c_r \tau\, N_t} = \mathrm{e^2 c_r \tau\, n_t\, LW} \quad (58)$$

n_t is the number of traps per unit area.

Substituting this into (55), we get:

$$\langle \mathrm{dI_{dssat}^2} \rangle_{1/f} = \left(\frac{3}{4\mathrm{n}}\frac{\mu}{\mathrm{L^2}}\,\mathrm{V_{gst}}\right)^2 \mathrm{e^2 LW\, c_r \tau \langle n_t \rangle} =$$
$$\mu \mathrm{e^2 c_r \tau \langle n_t \rangle} \frac{1}{\mathrm{L^2} n \mathrm{C'_{ox}}} \left(\frac{3}{4}\right)^2 2 \left(\frac{1}{2\mathrm{n}}\,\mu\,\mathrm{C'_{ox}}\,\frac{\mathrm{W}}{\mathrm{L}}\,\mathrm{V_{gst}^2}\right) \sim \frac{\mu \mathrm{e^2 c_r \tau \langle n_t \rangle I_{dssat}}}{\mathrm{L^2} n \mathrm{C'_{ox}}} \quad (59)$$

The relative change in current gets smaller when L is increased.

A similar equation also applies for current-mismatch:

$$\langle \mathrm{dI_{dssat}^2} \rangle_{\mathrm{mismatch}} \sim \frac{\mu \mathrm{e^2 c_p \langle n_t \rangle I_{dssat}}}{\mathrm{L^2 n C'_{ox}}} \quad (59)$$

Instead of the product of the capture rate and time ($c_r\ \tau$), we have used the capture probability c_p in eq. 59.

Discussion

In order to minimize 1/f noise and mismatch, the gate length should be increased. In order to keep V_{dssat} unchanged, the gate width (W) must be scaled up as well.

We can model the 1/f noise or mismatch with a current source at the transistor drain. We can also move the noise/offset source to the transistor gate. The following applies:

$$\langle dV_{gs}^2 \rangle = \frac{\langle dI_{ds}^2 \rangle}{g_m^2} \sim \frac{\mu e^2 \langle n_t \rangle I_{dssat}}{L^2 n C'_{ox} g_m^2}$$

Let us note that

$$g_m = \frac{1}{n} \mu C'_{ox} \frac{W}{L} \left(V_{gs} - V_{th}\right) = \sqrt{2 \frac{W}{L} \mu C'_{ox} I_{dssat}}$$

It follows:

$$\langle dV_{gs}^2 \rangle \sim \frac{\mu e^2 \langle n_t \rangle I_{dssat}}{L^2 n C'_{ox} g_m^2 f} \sim \frac{e^2 \langle n_t \rangle}{2LWnC'^2_{ox}}$$

Transistors with a large gate area have good matching and small 1/f noise.

From the book of Razavi (Design of Analog CMOS Integrated Circuits):

It is therefore not surprising to see devices having areas of several hundred square microns in low-noise applications.

PMOS devices exhibit less 1/ f noise than NMOS transistors because the former carry the holes in a “buried channel,” i.e., at some distance from the oxide-silicon interface, and hence trap and release the carriers to a lesser extent.

A high-pass filter or capacitive coupling makes a circuit less sensitive to 1/f noise or transistor mismatch. Using a high-pass filter is similar to a double measurement, where first a base-line measurement is performed, then a signal measurement is made, and then the difference is calculated. Such differential measurements are less affected by fluctuations caused by 1/f noise or transistor mismatch.

Power spectral density

The formula (58) corresponds to the integral of the power spectral density:

$$\langle dI_{dssat}^2 \rangle_{1/f} = \int_0^{\infty} S(f)\, df$$

How large is S(f)? We need to know S(f) to calculate noise when a transistor is used in a circuit.

The following equation holds: (DAS_2023_Rauschen_Theorie)

$$\boxed{S(f) = 2\frac{<Q^2>}{\Delta t} |\, P_{in}(i\omega)|^2 \quad (60)}$$

Let us first calculate the power spectral density for the traps with the time constant τ to $\tau+\Delta\tau$.

If we consider 1/f noise, Q for eq. 60 corresponds to the trapped amount of charge eΔN in time Δt. The following applies to the variance $<q^2>$:

$$\langle Q^2 \rangle = e^2 \langle \Delta N^2 \rangle = \text{Poisson} = e^2 \langle \Delta N \rangle = e^2 c_r \langle N_{t,\tau} \rangle \Delta t$$

$N_{t,\tau}$ is the number of traps with time constant τ to $\tau+\Delta\,\tau$. The following applies:

$$N_{t,\tau} = \Delta\tau\, N_t$$

where N_t is the total number of traps.

The trapped charge carriers are released with a time constant τ. Therefore:

$$Q(t) = Q\, e^{-\frac{t}{\tau}}$$

This leads to the current pulse:

$$I_{dssat}(t) = k\, Q\, e^{-\frac{t}{\tau}} \quad (61)$$

The pulse response Pin is defined as the output signal caused by the amount of charge of Q = 1C. Because of this:

$$P_{in}(t) = k\, e^{-\frac{t}{\tau}}$$

This function has the following Fourier transformation:

$$P_{in}(i\omega) = \frac{k\tau}{1+i\omega\tau}$$

The power spectral density of the current is then:

$$S_{I,\tau}(f) = 2\frac{<Q^2>}{\Delta t} |\, P_{in}(i\omega)|^2 = 2e^2 c_r \langle N_{t,\tau} \rangle \frac{\tau^2 k^2}{1+\omega^2\tau^2} \quad (62)$$

If we calculate integral (60) we get formula (58).

Equation 62 describes only the traps with the time constant τ. We can assume that for all traps the time constants are in a wide range: τ_{min} to τ_{max}.

The total power density for all traps is then:

$$S_I(f) = \int_{\tau_{min}}^{\tau_{max}} S_{I,\tau}(f) = 2e^2 c_r k^2 \int_{\tau_{min}}^{\tau_{max}} \langle N_t \rangle \frac{\tau^2}{1+\omega^2\tau^2}\, d\tau \sim \frac{2e^2 \langle N_t \rangle k^2}{\omega} \sim \frac{\mu e^2 \langle n_t \rangle I_{dssat}}{L^2 n C_{ox}' f} \quad (63)$$

We can move the noise source to transistor gate. The following then applies:

$S_V = \frac{S_I}{g_m^2} \sim \frac{\mu e^2 \langle n_t \rangle I_{dssat}}{L^2 n C'_{ox} g_m^2 f}$ (64)

Since

$$g_m = \frac{1}{n} \mu C'_{ox} \frac{W}{L} \left(V_{gs} - V_{th}\right) = \sqrt{2 \frac{W}{L} \mu C'_{ox} I_{dssat}}$$

we get:

$$S_V \sim \frac{\mu e^2 \langle n_t \rangle I_{dssat}}{L^2 n C'_{ox} g_m^2 f} \sim \frac{e^2 \langle n_t \rangle}{2LWnC'^2_{ox} f} \qquad (65)$$

We write it simplified:

$S_V = \frac{k_{1/f}}{f}$ (66)

$$k_{1/f} \sim \frac{\mu e^2 \langle n_t \rangle I_{dssat}}{L^2 n C'_{ox} g_m^2 f}$$

1/f noise output of the charge sensitive amplifier

Let us now calculate the contribution of the 1/f noise to the noise signal at the output of the charge sensitive amplifier. We can use the transfer function for the V_T.

$$H(i\omega) = \frac{V_o}{V_T} = \frac{(1+i\omega T_Z)}{(1+i\omega T_r)(1+i\omega T_f)} \qquad (60)$$

with:

$$T_r = \frac{C_i C_o + C_i C_f + C_f C_o}{C_f g_m}$$

$$T_f = R_f C_f$$

$$T_z = R_f (C_f + C_i^+)$$

1/f noise contribution is given by the integral:

$$< v_{o1/f\,T} >^2 = \int_0^\infty S_{V1/fT} |H(f)|^2 \, df \quad (66)$$

The frequency range between $1/T_z$ and $1/T_f$ contributes the most to the integral. In this frequency range, we can simplify the formula (66):

$$< v_{o1/f\,T} >^2 = k_{1/f} \int_{1/T_z}^{1/T_f} \frac{\omega T_Z^2}{1+\omega^2 T_f^2} d\omega = \frac{k_{1/f}}{2} \frac{T_Z^2}{T_f^2} \ln(1 + \omega^2 T_f^2) \,|^{1/T_f}{}_{1/T_z} \sim k_{1/f} \frac{(C_f + C_i^+)}{C_f^2}$$

Interestingly, the integral does not depend on time constants when the ratio T_z/T_f is constant! For smaller time constants, $S_{1/f}$ decreases but the frequency range in the integral ($1/T_z$ to $1/T_f$) increases.

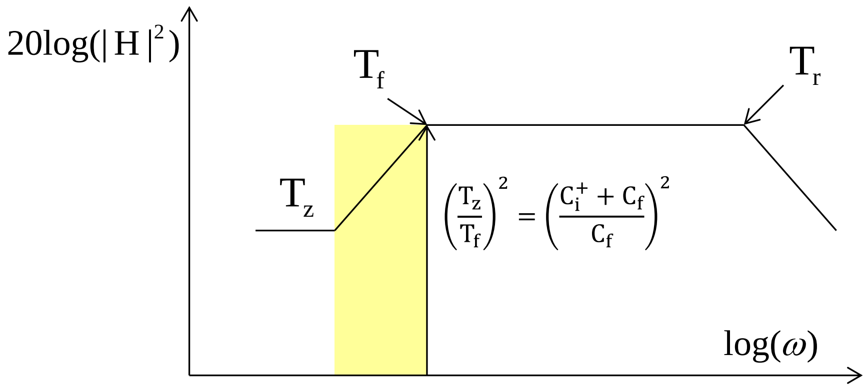


Figure 22: Frequency range between $1/T_z$ and $1/T_f$ contributes most to the integral

The variance of the noise signal at the output of the charge amplifier then becomes:

$$< v_o >^2 = \frac{1}{4}\frac{1}{C_f^2}\left((C_i^+ + C_f)^2\left(\frac{S_{VT}}{T_r} + k_{1/f}\right) + S_{IR}T_f + S_{ID}T_f\right)$$

The power spectral densities are:

$$S_{VT} = \frac{4kTn2/3}{g_m}$$

(thermal noise)

$$S_{IR} = \frac{4kT}{R_f}$$

(thermal noise)

$$S_{ID} = 2eI_{leak}$$

(leakage current noise)

$$k_{1/f} \sim \frac{\mu e^2 \langle n_t \rangle I_{dssat}}{L^2 n C'_{ox} g_m^2 f} \sim \frac{e^2 \langle n_t \rangle}{2LWnC'^2_{ox}}$$

(1/f noise)

When the detector leakage current is small and the time constant T_r is large, the 1/f noise dominates. W and L of the transistor should then be increased. A PMOS input transistor has smaller constant $k_{1/f}$.

Optimization for small 1/f noise can be done as the last step in the design.

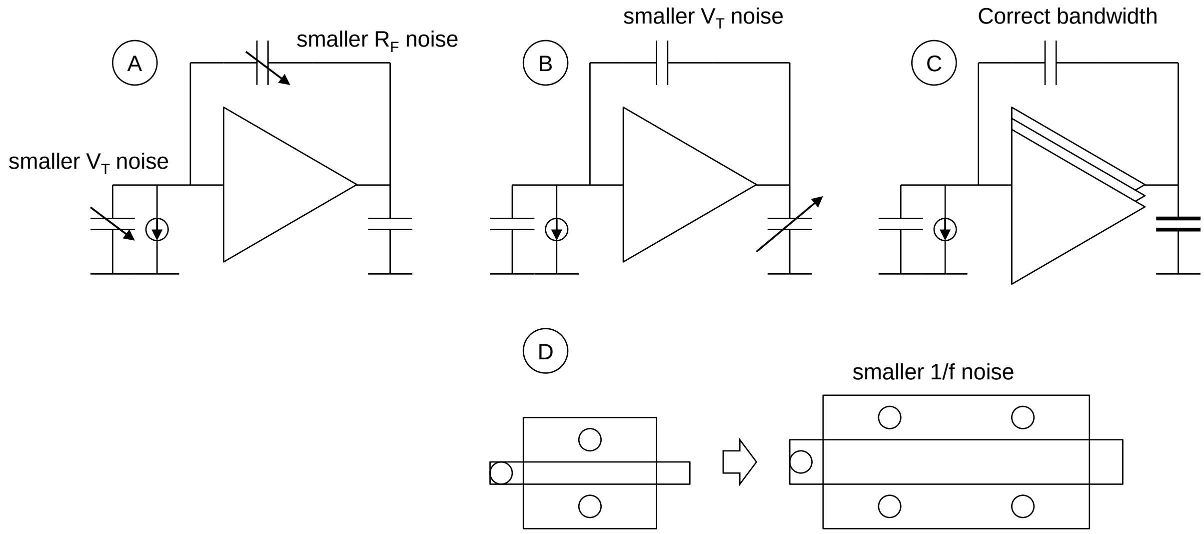


Figure 23: Optimization

Summary

Figure 24, Figure 25 and Figure 26 illustrate different noise sources.

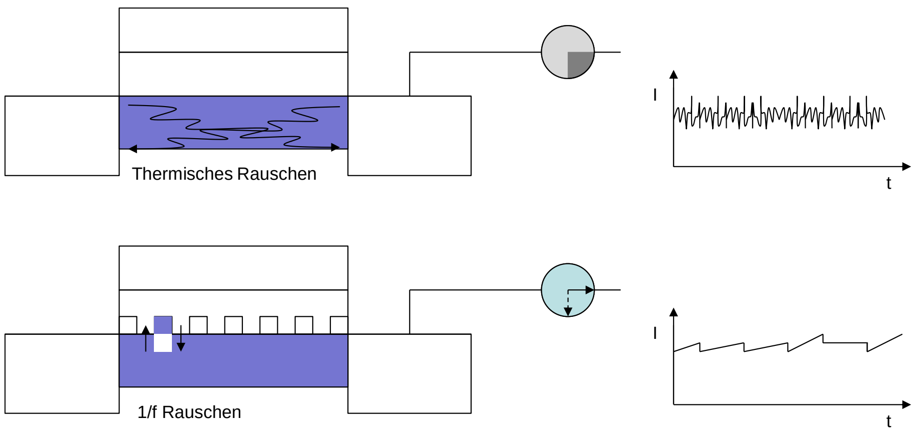


Figure 24: Noise in the transistor

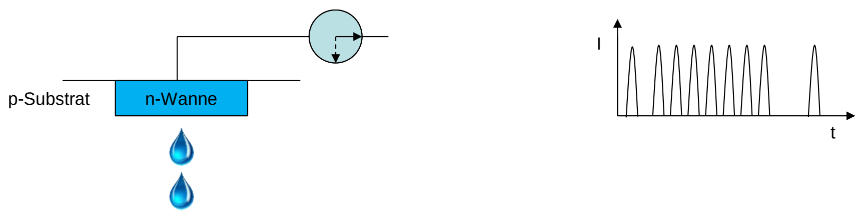


Figure 25: Sensor leakage current noise

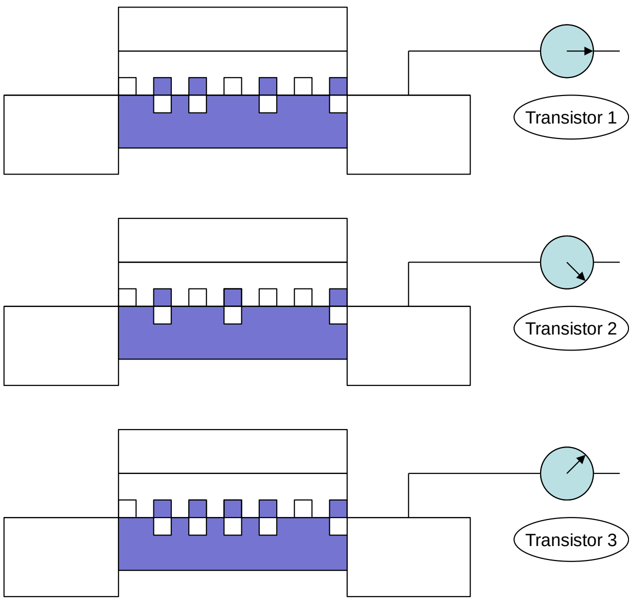


Figure 26: Transistor mismatch